C. N. WUNDER, University of Mississippi.
C. C. WYLIE, University of Iowa.
JESSICA M. YOUNG, Washington University.
E. I. YOWELL, University of Cincinnati.
C. M. LOWRY, Official Representative, John B. Stetson University.

The sessions of the American Association for the Advancement of Science began on Monday evening with the opening session in the Nashville War Memorial Building. After welcoming addresses by Dr. J. H. Kirkland, Chancellor of Vanderbilt University, and Judge Grafton Greene, Chief Justice of the Supreme Bench of Tennessee, Dr. A. A. Noyes, president of the American Association, gave an extended address in reply to the addresses of welcome. The retiring president, Dr. L. H. Bailey, was unable to be present because of serious illness but a letter from him was read at that time; this was followed by an illustrated lecture by Dr. S. G. Morley on "Recent excavations in Yucatan and Guatemala." A number of semi-popular lectures were given during the week under the auspices of the American Association, including an address on "E. E. Barnard, His life and work" by Dr. R. G. Aitken, of the Lick Observatory. On Wednesday afternoon Professor E. W. Brown of Yale University gave the fifth Josiah Willard Gibbs Lecture under the Auspices of the American Mathematical Society as a general lecture of the American Association; his subject was "Resonance in the solar system." This address is to appear in the BULLETIN of the American Mathematical Society.

Professor R. C. Archibald, Brown University, was chosen vice-president of Section A of the American Association. Professor C. N. Moore, University of Cincinnati, was elected secretary of the section, and Professor E. T. Bell, California Institute of Technology, a member of the section committee.

The mathematicians were housed in the dormitories of Ward-Belmont School in very comfortable fashion. They had their meals in the dining-room of the School, and the parlors afforded very convenient meeting places for general social purposes. Nearly two hundred attended the joint dinner of the mathematicians which was held Thursday evening under the eloquent toastmastership of Professor Archibald Henderson. Professor Snyder as president of the Society spoke on the esthetic side of mathematics as developed by the activities of the Society. Professor Slaught as a representative of the Mathematical Association spoke of the men prominent in the history of the Monthly and the Association, of the part in the development of expository work which is possible from the Association publications, and of the arousing of active interest in the many groups of its sections over the country. Miss Jewell Hughes told of the opportunities for women in the teaching of mathematics in the United States, affirming and substantiating the statement that women have an opportunity equal with that of men in college and university positions in mathematics unless it be in the higher university positions. Dr. C. F. Roos

told of the National Research Fellowships with their opportunity for uninterrupted studies, largely under one's own initiative but identifying one's work with some institution; he spoke particularly of the benefit which comes from a general discussion of one's problem with members of the faculty and of desirable help upon the bibliography of the subject. Professor R. M. Winger of the University of Washington spoke as a representative of the Western mathematicians and told of various notable accomplishments in that region. Professor R. L. Moore of the University of Texas spoke as a representative of the higher institutions of learning in the South. After raising the question whether it is desirable to extend the number of departments of mathematics in the South conducting extensive graduate work and hence conferring the degree of Ph.D., he concluded that there may very well be departments in the South that are outstanding in some particular field and that may discover young men of talent who would otherwise be unknown, and expressed his belief that the number of students going North for their doctor's degrees may well be increased. Professor J. L. Coolidge who spoke "for mathematicians everywhere" said that we come to these meetings to meet kindred spirits and to bring together the teachers from the larger and smaller colleges to consider our common ideals; our highest usefulness is to maintain the ethical etiquette and standards of our students, to meet the increasing competition between the best brains in America and Europe with the highest honesty and with warm, sympathetic cooperation.

The convenience of the arrangements and the many pleasant features which surrounded the stay in Ward-Belmont School were recognized by a motion offered by Professor Rietz and adopted by a rising vote at the last session, whereby the Association expressed its thanks to the Committee on Arrangements under the chairmanship of Professor Miser and in particular to the authorities of Ward-Belmont School for the hospitality which was extended during the sessions.

The American Mathematical Society held its thirty-fourth annual meeting on Tuesday morning and afternoon and Wednesday morning for the reading of papers. At the beginning of the Wednesday morning session Professor James Pierpont gave an address on "Mathematical rigor, past and present" by invitation of the Society; this address appears in the Bulletin of the Society for January-February 1928.

The program of the sessions beginning on Thursday afternoon is given herewith, accompanied by abstracts of part of the papers. The program was prepared by the following committee: Professors David Barrow, H. J. Ettlinger, S. W. Reaves, and H. E. Buchanan, Chairman.

JOINT SESSION OF THE ASSOCIATION WITH THE SOCIETY AND SECTION A OF THE AMERICAN ASSOCIATION

1. "The notion of probable error in elementary statistics" by Professor E. V. HUNTINGTON, Harvard University, Retiring vice-president of Section A of the American Association.

2. "The human significance of mathematics" by Professor DUNHAM JACKSON University of Minnesota, Retiring president of the Mathematical Association.

3. "Some philosophic aspects of mathematics" by Professor ARNOLD DRESDEN, Swarthmore College, representing the Mathematical Society.

Abstracts are not given for these three addresses for the reason that they are published in full in journals readily available to any reader. The address by Professor Huntington appeared in Science for Dec. 30, 1927; that by Professor Jackson will appear in an early issue of the Monthly and that by Professor Dresden in an early issue of the Bulletin of the American Mathematical Society.

FIRST SESSION OF THE ASSOCIATION

1. "Observations on simultaneous equations" by Professor ARCHIBALD HENDERSON, University of North Carolina.

2. "Mathematics in the junior colleges" by Professor JEWELL C. HUGHES, University of Arkansas.

3. "The reorganization of secondary mathematics in theory and practice" by Vice-principal WILLIAM BETZ, East High School, Rochester, New York.

Abstracts of these papers follow:

1. The usual methods employed for the solution of simultaneous quadratic equations, to which subject the present paper is restricted, lead to the solution of a quartic equation in either variable, which lends itself to solution according to familiar methods. Equations of the type $x^2+y=a^2+b$, $x+y^2=a+b^2$, where a and b are rational, may however be reduced to the solution of a cubic representing the points of intersection in either coordinate of either of the original parabolas with the hyperbola $(x+a)(y+b)-1=0$.

In the present paper a general method, depending upon the solution of a cubic equation, is presented for effecting the solution of the pair of quadratic equations in two variables, in the most general form:

$$U_1 \equiv a_0x^2 + a_1y^2 + 2a_2xy + 2a_3x + 2a_4y + a_5 = 0, \tag{1}$$

$$U_2 \equiv b_0x^2 + b_1y^2 + 2b_2xy + 2b_3x + 2b_4y + b_5 = 0. \tag{2}$$

Employing homogeneous coordinates, we seek the condition that the conic through the four points of intersection of (1) and (2), viz., $u_1+\lambda u_2=0$ reduce to a pair of lines. The solutions of the cubic

$$\begin{vmatrix} a_0 + \lambda b_0 & a_2 + \lambda b_2 & a_3 + \lambda b_3 \\ a_2 + \lambda b_2 & a_1 + \lambda b_1 & a_4 + \lambda b_4 \\ a_3 + \lambda b_3 & a_4 + \lambda b_4 & a_5 + \lambda b_5 \end{vmatrix} = 0,$$

give rise to three pairs of lines $p_1=0$, $p_2=0$; $q_1=0$, $q_2=0$; $r_1=0$, $r_2=0$. The solutions of equations (1) and (2) are then $(p_1q_1r_1)$, $(p_1q_2r_2)$, $(p_2q_1r_2)$, $(p_2q_2r_1)$. All the different cases, depending upon particular collocations of the lines, in the progressive degeneration of the complete quadrangle, are analyzed. The method outlined is applicable to all cases, save the trivial one when all four intersections are coincident, which point is obviously the point of tangency of the common tangent to $U_1=0$, $U_2=0$, at the intersection.

2. The address by Miss Hughes will be published in the Monthly.

In the discussion which followed Professor Slaught spoke of the importance to teachers of junior college mathematics of their becoming members of the Association and of our having the cooperation of members of the Association in inviting to membership those in our immediate neighborhoods. Professor Rietz said that the question with regard to credit to be given to work in junior colleges by the four-year colleges and universities is as yet vague and must be controlled by proper inspection of the junior colleges, and by a knowledge of the competence of the teacher and the character of the course. Professor T. M. Simpson urged that the state representatives present carry the cause of the Association to the junior colleges in our own states.

3. After a brief analysis of the present educational situation, Mr. Betz showed that a crisis has resulted from the conflicting views of specialists and educational theorists. It is of paramount importance, in his opinion, that these opposing groups should work in greater harmony toward a tangible goal. The speaker offered a critical review of the progress made in reorganizing secondary mathematics during the past quarter of a century. He pointed out that neither the hopes of the reformers of twenty-five years ago nor the demands of the general educator had been fully realized. The compartment system is still in full vogue, and the "unification of pure and applied mathematics" is still a dream. The curriculum expert is complaining of the purely "academic" character of mathematical instruction and of the total disregard of many necessary readjustments. Mr. Betz insisted that more attention must be given to the human significance of mathematics, to scientific classroom procedures, and to improved professional resources for the training of teachers. In conclusion, the speaker offered a tentative six-year curriculum. He also submitted to the Association a memorandum on behalf of the curriculum committees appointed by the New York State Department of Education, asking for cooperation in the preparation of new syllabi for New York State.

In the discussion of this paper, Professor Slaught said that we offer no apology for our interest in secondary mathematical education; that one of our

first plans was to form the National Committee, and that it is perfectly logical that we continue our active interest in the improving of the teaching of mathematics. He commended, too, the yearbooks of the National Council of Teachers of Mathematics, the mathematical federation of the hundred groups which were brought together by the National Committee. (The Trustees adopted a resolution on the cooperation requested by Mr. Betz, as is indicated in the account of the meeting of the Trustees). Professor Lytle asked why we do not have in America the same cooperation and active interest in the secondary associations which are given by eminent mathematicians in England in connection with the Mathematical Association of England. Professor Lane replied that our Mathematical Association represents exactly the answer to Dr. Lytle's question, for many of the men in America interest themselves sympathetically in secondary education and in the training of teachers even though they are themselves research men. As an instance of the sort there was mentioned the joint session and the joint meetings of such organizations as the Louisiana-Mississippi Section and the branch of the National Council.

SECOND SESSION OF THE ASSOCIATION

1. "The fundamental mathematical requirements of Biology" by Professor J. ARTHUR HARRIS, Head of the Department of Botany, University of Minnesota. (By title.)

2. "Present tendencies in projective geometry" by Professor E. P. LANE, University of Chicago.

3. "The law of small numbers" by Professor A. R. CRATHORNE, University of Illinois.

4. "Application of vectors to the formation of the roots of the cubic" by Professor J. W. LASLEY, Jr., University of North Carolina.

Abstracts of the papers follow:

1. On account of Professor Harris's unavoidable absence from the meeting, his paper was read by title only, but it will be published in the Monthly, much to the gratification of a considerable number who expressed their interest in this special topic.

2. The present tendencies in projective geometry can best be understood after a brief survey of the origin and development of this science. Although there existed isolated instances of theorems of a projective nature before 1800, still there was no organized projective theory before the fundamental work of Poncelet in the first quarter of the nineteenth century.

Synthetic projective geometry grew most rapidly in the first half of the last century, and analytic projective geometry about the middle. In the latter part of the century Halphen began the first systematic projective differential investigations.

In the first quarter of the present century, Wilczynski founded the American school of projective geometry, and Fubini founded an Italian school, using very different methods.

The present tendency is to correlate the two methods used by Americans and Italians, and to attempt to develop a systematic theory of the projective differential geometry of a general variety in hyperspace.

3. Professor Crathorne's paper is an expository and critical discussion of the "Law of Small Numbers," giving its history, its application in statistics, and a resumé of the objections which have been raised against it.

Professor Rietz in commenting upon the objections made to the name "Bortkiewitsch's Law of Small Numbers," said that the authors of the *Handbook of Mathematical Statistics* chose this term as against the term "Poisson's Formula" supported by certain French mathematicians. Mr. Molina stated that the Bell Telephone Company is determining the number of trunks required for definite forms of service by the use of this formula.

4. This paper concerns itself with two pedagogical questions relative to the general solution of the cubic equation: first, the solution itself as to technic, second, the formation of the solution from the roots of the resolving quadratic. A solution is here presented based on the binomial theorem and the relation between the roots and coefficients of a quadratic equation. It is believed to be free from the artificiality and arbitrariness of the usual text-book solution of the cubic. The notion of vectors is applied to the formation of the roots of the cubic from those of the resolving quadratic. For this purpose vectors called "symmetric" are defined. Two vectors are said to be symmetric if the sum of their angles is an integral multiple of π. Certain properties of symmetric vectors are deduced, and on the basis of these a method is outlined for forming by means of the symmetric vectors the roots of the cubic from those of the quadratic. This method, it is believed, will impress the student with the fact that one need not always begin with the principal cube root of the roots of the resolving quadratic, but may begin the formation at will with any cube root of either root of that equation. Moreover, the solution is obtainable with equal ease whether or not one begins with the principal cube roots.

Meetings of the Board of Trustees

Eleven members of the Board of Trustees were present at the sessions.

The following twenty persons and five institutions were elected to membership on applications duly certified:

To Individual Membership

Eugene Alliot, Lic. ès Sc. (Paris). Prof., St. Michael's Coll., Winooski, Vt.

O. K. Bower, A.M. (Illinois). Asst. Instr., Univ. of Illinois, Urbana, Ill.

Mary S. Day, Ph.D. (Columbia). Head of Dept., Salem Coll., Winston-Salem, N.C.

Gordon Fuller, A.B. (W. Tex. State Teachers Coll.). Grad. Student, Univ. of Michigan, Ann Arbor, Mich.

GERTRUDE A. HERR, M.S. (Iowa State Coll.). Asso. Prof., State Coll., Ames, Ia.

R. E. HILL, A.M. (Louisville). Head of Dept., Univ. of Louisville, Louisville, Ky.

LYLAH KRYDER, A.M. (Columbia). Instr., Hunter Coll., New York, N. Y.

W. N. MEBANE, JR., B.S. (Davidson). Asst. Prof., Davidson Coll., Davidson, N. C.

W. L. MURFEE, Col., A.M. (Virginia). Pres., Marion Inst., Marion, Ala.

W. B. ORANGE, A.M. (California). 5071 Whittier Blvd., Los Angeles, Calif.

LELA OXSHEER, A.M. (Teachers Coll., Columbia). Asso. Prof., Stephen F. Austin Teachers Coll., Nacogdoches, Tex.

W. O. PENNELL, B.S. in E.E. (Mass. Inst. of Tech.). Chief Engineer, So. Western Bell Tell. Co., St. Louis, Mo.

J. F. RANDOLPH, B.S. (W. Tex. State Teachers Coll.). Grad. Student, Univ. of Michigan, Ann Arbor, Mich.

H. H. RAPHAEL, A.M. (California). Instr., Jr. Coll., Glendale, Calif.

HERBERT REBARKER, Ph.D. (Peabody). Head of Dept., Demonstration School, and Teacher of the Teaching of Math., Peabody Coll., Nashville, Tenn.

H. P. ROGERS, A.M. (Illinois). Instr., James Millikin Univ., Decatur, Ill.

I. D. STEWART, A.B. (Muskingum). Instr., Bryson Coll., Fayetteville, Tenn.

W. H. TAYLOR, Ph.D. (Iowa). Registrar and Prof. Hamline Univ., St. Paul, Minn.

L. W. TURNER. Supervisor Agency, Statistical Dept., Hartford Acc. and Ind. Co., Hartford, Conn.

C. H. WHEELER III, B.S. (Washington and Jefferson). Student Asst., Johns Hopkins Univ., Baltimore, Md.

G. T. WHYBURN, Ph.D. (Texas). Adj. Prof., Pure Math., Univ. of Tex.

To Institutional Membership

ALABAMA COLLEGE, Montevallo, Ala.

JOHN B. STETSON UNIVERSITY, DeLand, Fla.

LOUISIANA STATE UNIVERSITY, Baton Rouge, La.

ST. JOHNS COLLEGE, Annapolis, Md.

NORTH TEXAS AGRICULTURAL COLLEGE, Arlington, Tex.

The following were appointed associate editors of the Monthly for the year 1928:

H. E. BUCHANAN	H. S. EVERETT	C. N. MILLS
ELIZABETH CARLSON	B. F. FINKEL	F. D. MURNAGHAN
OTTO DUNKEL	R. A. JOHNSON	H. L. OLSON
H. J. ETTLINGER	H. W. KUHN	D. E. SMITH

The financial statement for the year 1927 was presented by Professor Cairns. The report had been examined by Professor Slaught for the Finance Committee and by Professor Crathorne for the Association. It was accepted and approved by the Trustees.

The Committee on Carus Monographs requested that any suggestions as to possible future monographs be given in writing to the committee and they requested that the committee be enlarged so as to number at least five. This appointment will be made by the president at an early date.

The By-Laws of the recently organized Philadelphia Section were approved.

As an expression of loyalty to the American Mathematical Society and of the unique relation which the Association bears to the Society, the Trustees voted to apply for a sustaining membership in the Society. Five members of the Society will be nominated under this membership.

An invitation was received from the president and the department of mathe-

matics of the University of Minnesota to hold the summer meeting there in 1931, in connection with a similar meeting of the Mathematical Society.

The Trustees adopted the following resolution:

"The Association recognizes the benefit accruing from membership in mathematics clubs and honor mathematical societies, and commends the favorable consideration of these to colleges and universities as a means of encouraging and stimulating students in the early part of their career and of furnishing a stepping stone toward membership in the Association and the society."

In response to the request of the New York State Syllabus Committee on Algebra, submitted by Mr. William Betz, for the Association's active cooperation in dealing with the program of requirements in algebra, the following resolution was adopted:

"Resolved that the Trustees of the Mathematical Association of America wish to express their interest in the efforts of the Algebra Syllabus Committee of the New York State Department of Education to revise its syllabus for the eleventh and twelfth years of the high school course in the spirit of the recommendations of the National Committee on Mathematical Requirements, and that a committee of three members of the Association be appointed to act in an advisory capacity when so requested by the Algebra Syllabus Committee."

Professor J. W. Young, E. R. Hedrick and Ralph Beatley were appointed as this committee.

The president was empowered to appoint delegates to the International Mathematical Congress.

The Finance Committee was empowered to transfer another thousand dollars to the General Endowment Fund if this seems wise.

Other matters were considered concerning the formation of sections of the Association and concerning possible provision for an extension of facilities for the publication of certain types of mathematical articles.

Annual Business Meeting of the Association

Professor Cairns reported the death of the following members:

R. F. Borden, Assistant professor of mathematics, George Washington University, Washington, D. C. (March 15, 1927).

W. G. Bullard, Professor of mathematics, Syracuse University (February 16, 1927).

F. H. Loud, Professor emeritus of mathematics, Colorado College (March 2, 1927).

G. A. Osborne, Professor emeritus of mathematics, Massachusetts Institute of Technology (November 20, 1920).

H. E. Russell, Professor of mathematics, University of Denver (May 31, 1927).

A. C. Bose, Deputy Magistrate, Bengal Provincial Civil Service, Calcutta, India (December 11, 1926).

C. D. Garlough, Professor of mathematics, Wheaton College (November 25, 1927).

R. L. Modesitt, Teacher, Eastern Illinois State Teachers College (December 16, 1927).

J. L. Riley, Professor of mathematics, Ouachita College (July 12, 1927).

H. D. Thompson, Professor of mathematics, Princeton University (August 30, 1927).

The election of officers for the year 1928 was conducted by mail and in

person at this meeting; the tellers, Professors J. D. Bond and G. W. Hess, reported the result of the balloting as follows:

For Vice-Presidents: A. J. Kempner, 253 votes; R. E. Moritz, 205 votes; F. D. Murnaghan, 251 votes; Miss Clara E. Smith, 217 votes.

For additional members of the Board of Trustees, to serve until January 1931: R. C. Archibald, 301 votes; D. Buchanan, 218 votes; L. P. Eisenhart, 303 votes; B. F. Finkel, 206 votes; J. R. Kline, 125 votes; E. P. Lane, 224 votes; W. L. Miser, 134 votes; H. L. Rietz, 332 votes.

The following were accordingly declared elected:

Vice-Presidents: A. J. KEMPNER, University of Colorado; F. D. MURNAGHAN, Johns Hopkins University.

Additional members of the Board of Trustees: R. C. ARCHIBALD, Brown University; L. P. EISENHART, Princeton University; E. P. LANE, University of Chicago; H. L. RIETZ, University of Iowa.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 12, 1927

RECEIPTS			EXPENDITURES		
Balance Dec. 16, 1926		$7,514.66	Publisher's bills (Nov. '26–Oct. '27)		$4,899.01
1925 indiv. dues	$ 87.00		Manager's office		32.01
1926 indiv. dues	364.70		Editor-in-Chief's office 1926		51.47
1926 instit. dues	31.00		Editor-in-Chief's office 1927		576.77
1927 indiv. dues	6,301.62		*Register* expense		95.75
1927 instit. dues	769.30		Committee on Membership		145.00
1927 subscriptions	681.75		Joint Com. on Membership		89.06
Initiation fees	280.00		Expense, Report, Com. on Math. Requirements		34.09
Advertising	778.95		Information Bur. for Appointments		8.50
Sale copies of MONTHLY	88.65		Secretary-Treasurer's office:		
Sale reprints	1.35		Postage	$383.20	
Sale First Carus Mon.	63.75		Bond	5.00	
Sale Second Carus Mon.	111.25		Ins. back issues MONTHLY	6.40	
Sale Third Carus Mon.	1,095.25		Safety Deposit	4.00	
From Mon. Fund for expenses	139.75		Office supplies	45.70	
Recd. for subscriber's Monographs	2.00		Express, tel., etc.	42.30	
For *Annals* subscriptions	3.00		Clerical work	528.59	
Recd. for expense La.–Miss. Section	18.00		Printing	436.50	
Life memb. payments	118.62		Library expense	7.25	
Sale Rhind Papyrus	1,020.00		Pd. copies MONTHLY	14.25	
From matured bond	500.00		Philadelphia meeting	80.00	
Contributions A. B. Chace	5,898.66		Madison meeting	69.41	
Int. Oberlin Savgs. Bk.	145.87		Nashville meeting	15.10	
Int. Peoples Bkg. Co.	121.67		Refd. subscriptions	17.10	
Int. Liberty Bonds	85.00		Refd. Third Mon.	1.25	
Int. Hardy Fund	120.00		Forwarded for subscribers' Monograph	2.00	
Int. on Chace contributions	40.20		Forwarded init. fee to *Math. Teacher*	2.00	
		18,867.34			1,660.05
Total 1927 receipts		$26,382.00			

		Annals subvention	250.00
		Pd. *Annals* subscriptions	4.50
		Forwarded *Annals* subscriptions	3.00
		Pd. to sections from initiation fees	175.52
		Cost of new bond	523.47
		Pd. int. Hardy Fd. to B. F. Finkel	120.00
		Honorarium Carus Monograph	300.00
		Transferred to Mon. Fd.	1,218.90
		Expense acct. Mon. Fd.	85.00
		Cost of bonds for Genl. Endowment Fd.	2,091.26
		Printing Rhind Papyrus	5,771.98
		Pd. expense La.-Miss. Sec.	18.00
Total expenditures	18,153.34	Total expenditures	$18,153.34
Balance to the end of 1927 business	$8,228.66	Cash on hand	4.00
Received on 1928 business	811.50	Checking account	584.66
		Oberlin Savgs. Bk. acct.	3,608.53
		Peoples Bkg. Co. acct.	1,754.11
		Liberty Bonds	1,000.00
		Certifs. of deposit	2,088.86
Book balance Dec. 12, 1927	$9,040.16	Bank balance Dec. 12, 1927	$ 9,040.16

Carus Monograph Fund

Balance Dec. 16, 1926 ($2,824.42; $201.40)		$3,025.82
Receipts: Sales	$1,270.25	
Interest	149.49	1,419.74
		$4,445.56
Expenditures: Honorarium	$ 300.00	
Expense account	85.00	385.00
		$4,060.56
Certificate of deposit	$4,053.06	
Cash in bank	7.50	
		$4,060.56

Arnold Buffum Chace Fund

Receipts from advance sales	$1,020.00
Balance Dec. 12, 1927, certificate of deposit	$1,020.00

Chauvenet Prize Fund

Balance Dec. 16, 1926	$20.00
Amount set aside for 1927	20.00
Balance Dec. 12, 1927, cash in bank	$40.00

Life Membership Fund

Liability on life memberships Dec. 16, 1926 (Corrected)	$336.65	
Receipts for life memberships	118.62	
		$455.27
To be transferred to current funds, surplus		11.43
Liability on life memberships Jan. 1, 1928		$443.84

GENERAL ENDOWMENT FUND

Gift from previous owners of the MONTHLY		$1,000.00
Transferred from current funds in 1926		1,000.00
Transferred from current funds in 1927		1,000.00
Interest, U.S. Treasury Notes		150.00
Interest, Liberty Bonds		42.50
		$3,192.50
U.S. Treasury Notes, present value	$1,150.00	
Liberty Bond	1,000.00	
Land Trust Certificate	1,000.00	
Cash in bank	42.50	
		$3,192.50

Of the funds on hand, as indicated above, $7.50 belongs to the CARUS MONOGRAPH FUND (not yet transferred); $1,020.00 belongs to the ARNOLD BUFFUM CHACE FUND; $126.68 remains for the expense of printing the Rhind Papyrus; $40.00 is set aside as the first two payments toward the Chauvenet Prize Fund; $443.84 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1928; $42.50 belongs as interest to the GENERAL ENDOWMENT FUND. Aside from these, the sum of $3,000 is held in reserve as a GENERAL ENDOWMENT FUND; and the CARUS MONOGRAPH FUND, now amounting to $4,053.06, is carried as a separate fund in the form of a certificate of deposit which bears 4%, compounded quarterly.

When the accounts were closed Dec. 12, 1927, there remained on the total business for the year 1927 the following items:

BILLS RECEIVABLE (partly estimated)		BILLS PAYABLE (partly estimated)	
1927 individual dues	$ 130.00	Publisher's bills (Nov., Dec. '27)	$1,150.00
Advertising	70.00	President's office	40.00
	$ 200.00	Manager's office	40.00
		Editor-in-Chief's office	100.00
		Other editors' postage	50.00
		Committee on Membership	50.00
		Register	600.00
		Secretary-Treasurer's office	250.00
		Annals subvention	200.00
		Initiation fees due to sections	600.00
		Carus Monograph Fund	7.50
		Chace Fund	1,020.00
		Rhind Papyrus funds	126.68
		Chauvenet Prize Fund	40.00
		Life Membership Fund	443.84
		General Endowment Fund	42.50
			$4,760.52

If to the balance on 1927 business shown in this report, $8,228.66, there be added the bills receivable, $200.00, and there be subtracted the estimated bills payable, $4,760.52, there results an estimated final balance on 1927 business of approximately $3,700, a satisfactory advance in the Association's resources for the year.

W. D. CAIRNS, *Secretary-Treasurer*

THE MECHANICAL COMBINATION OF LINEAR FORMS

By D. H. LEHMER, University of California, Berkeley.

The "method of exclusion" introduced by Gauss, although a tentative method, is still one of the most powerful weapons for attacking many problems in the theory of numbers. Perhaps the most familar of such problems is the solution of the congruence $x^2 \equiv D \pmod{m}$. The method is also of great use in the problem of representing a number by a given form such as

$$a^2 - b^2, \quad a^2 + b^2, \quad a^2 \pm Db^2, \cdots .$$

Suppose, for instance, we wish to represent a given number N as the difference of two squares. We may choose 5 as an "excluding number." Let $N \equiv 3 \pmod 5$. We construct the following table

$$\begin{aligned} a &\equiv 0\ 1\ 2\ 3\ 4 \\ a^2 &\equiv 0\ 1\ 4\ 4\ 1 \pmod 5 \\ a^2 - N = b^2 &\equiv 2\ 3\ 1\ 1\ 3 \end{aligned}$$

But 2 and 3 are non-residues of 5 so that the cases $a \equiv 0$, 1, and 4 (mod 5) are excluded. We have then $a = 5n+2$, 3. Other small prime moduli may be used as excluding numbers with the result that a is restricted to $(p \pm 1)/2$ values modulo p. The problem that now presents itself is the combination of the linear forms thus obtained. If the range of a is large, it is necessary to use many excluding numbers and the labor of combining directly these linear forms is prohibitive.

A graphical method has been suggested[1] to solve this difficulty in which use is made of a table ruled in squares, each cell representing a possible value of a. "Movable strips" are made, the length of each being some excluding number p, and on which impossible values of a modulo p are indi-

[1] Kraitchik, *Theorie des Nombres* (Paris, 1922), Chapter 2.

cated by crosses. These strips are moved down the columns of the table and the cells adjacent to the crosses on the strip are then ruled out. As many strips as are necessary are computed and applied until the number of empty cells is reasonably small for actual trial.

But even this method has certain disadvantages. In the first place it requires very close attention and much care. Also the strips are short in comparison with the length of the table and have to be shifted frequently, which is a possible source of error. Moreover the method makes it necessary to fix the limit of the table before commencing the work, so that the whole range must be entirely covered before any definite information can be obtained.

These difficulties are overcome by a machine constructed by the writer for the combination of linear forms. It is the object of this paper to give a brief description of the machine and to exhibit some of the results obtained by it.

Here instead of strips we have chains. The number of links in each chain is a convenient multiple of some small prime. In fact the chain lengths are: 64, 27, 25, 49, 22, 26, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, and 67. Each chain hangs in a loop from a sprocket having 10 teeth. All the 19 sprockets are fixed rigidly to a single shaft driven by a motor. The possible values modulo p are indicated by small pins fastened to the appropriate links of the chain (the zero link being indicated by red paint.) Whenever a link provided with a pin arrives at the top of the shaft a small spring with a tungsten point is lifted by the projecting pin. This breaks for the moment the electric contact between the spring and a brass bar running parallel to the shaft. By means of a relay in the circuit, the motor is shut off and the machine stops itself. When several chains are provided with springs the machine will not stop unless all the springs are lifted, so that every time the machine stops it means that a number satisfying all the imposed conditions has appeared. This number can be read directly by means of a revolution counter connected to the shaft. The shaft revolves 300 r. p. m. so that the machine canvasses 3000 numbers per minute. When all chains are provided with springs a "solution" occurs once in several hours during which time the machine runs without any attention.

The use of the machine is best explained by means of examples. Let us first consider the representation of

$$N = (10^{20} + 1)/(10^4 + 1) = 9999000099990001 = a^2 - b^2$$

as the difference of two squares. A representation besides

$$a \equiv (N + 1)/2, \quad b = (N - 1)/2$$

is possible since N has been shown to be the product of two primes.[1] The factors of N are of the form $40n+1$. The expansion of the square root of N in a regular continued fraction seems to indicate that, although 21 is a residue of N, 3 and 7 are both non-residues. The assumption that 3 is a non-residue gives the following form for the factors of N:

$$\left.\begin{matrix} 3n+2 \\ 40n+1 \end{matrix}\right\} \quad 120n+41.$$

Let us then write

$$a+b = 120n+41, \tag{1}$$

$$a-b = 120m+41. \tag{2}$$

Multiplying (1) and (2) we get

$$a^2-b^2 = N = 120^2mn + 120\cdot 41(m+n) + 41^2.$$

Now $N \equiv 10801 \pmod{120^2}$. Therefore $120\cdot 41(m+n) \equiv 9120 \pmod{120^2}$, or

$$41(m+n) \equiv 76 \pmod{120}.$$

Solving this congruence we have

$$(m+n) \equiv 116 \pmod{120}. \tag{3}$$

Adding (1) and (2) we get $a=60(m+n)+41$. Substituting from (3) we get

$$a = 60(120k+116)+41 = 7200k+7001. \tag{4}$$

The range for a is given by the formula

$$\sqrt{N} < a < \frac{1}{2}\left(W+\frac{N}{W}\right),$$

where W is the limit to which the number has been searched for factors. In this case[2] $W=120{,}000$ so that a has the range

$$99995000 < a < 41662560416 \text{ with } 13888 < k < 5786466.$$

Now let $x=k-13888$ so that $0<x<5772578$. Then (4) gives

$$a = 7200(x+13888)+7001 = 7200x+100000601, \tag{5}$$

so that a is restricted to one case in 7200.

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[1] Bulletin of the American Mathematical Society, vol. 33, p. 336.

[2] This limit has been established by Shanks' table of the number of digits in the repetend of the reciprocal of every prime<120000.

We proceed now to exclude x with small prime moduli. Let us take for example the excluding number 11. We have from (5)

$$a \equiv 6x + 8 \pmod{11} \qquad N \equiv 1 \pmod{11}.$$

We now construct the following table:

$$\begin{aligned} x &\equiv 0\ 1\ 2\ 3\ \ 4\ 5\ \ 6\ 7\ 8\ 9\ 10 && \pmod{11}\\ a \equiv 6x+8 &\equiv 8\ 3\ 9\ 4\ 10\ 5\ \ 0\ 6\ 1\ 7\ \ 2\\ a^2 &\equiv 9\ 9\ 4\ 5\ \ 1\ 3\ \ 0\ 3\ 1\ 5\ \ 4\\ a^2 - N = b^2 &\equiv 8\ 8\ 3\ 4\ \ 0\ 2\ 10\ 2\ 0\ 4\ \ 3 \end{aligned}$$

Excluding those values of x which correspond to non-residues in the last line, we get the following linear forms for x modulo 11:

$$x = 11n + 2,\ 3,\ 4,\ 8,\ 9,\ 10.$$

Other excluding numbers may be dealt with in like manner.

Kraitchik[1] has given tables of the possible values of a for any N and for excluding numbers $\leqq 47$. The possible values of x can easily be obtained from the tabulated values of a by means of the transformation

$$x \equiv \frac{1}{k}a - \frac{1}{k}l \pmod{p} \text{ whenever } a \equiv kx + l \pmod{p}.$$

These tables have been recalculated and the errors noted are given at the end of the present paper. In order to serve the needs of the machine these tables have been extended to excluding numbers $\leqq 67$. The use of the tables is to be preferred to the method illustrated by the above example. The calculation is much simpler and the results can easily be checked by symmetry. For instance the table for 11 gives in our case

$$a \equiv 1,\ 2,\ 4,\ 7,\ 9,\ 10 \pmod{11}.$$

Making the transformation

$$x \equiv \frac{1}{6}a - \frac{1}{6}8 \equiv 2a + 6 \pmod{11},$$

we get $x \equiv 8, 10, 3, 9, 2, 4 \pmod{11}$ as before.

We can also make good use of the condition that 7 is a non-residue of N. The factors of N are then of the form $7n+3$, 5, or 6. Since $N \equiv 2 \pmod{7}$, we have the following two cases to consider.

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[1] Kraitchik, loc. cit., pp. 187–199.

CASE I. $a+b=7n+3$; $a-b=7m+3$.

$$N = a^2 - b^2 = 49mn + 21(m+n) + 9 \equiv 9 \pmod{7^2}.$$

$$\therefore m+n \equiv 0 \pmod 7 \; ; \quad a \equiv 7\left(\frac{m+n}{2}\right) + 3 \; ; \quad a \equiv 3 \pmod{49}.$$

CASE II. $a \pm b = 7n+5$; $a \mp b = 7m+6$.

$$N = 49mn + 7(5m+6n) + 30 \equiv 9 \pmod{7^2}.$$

$$\therefore 5m+6n \equiv 4 \pmod 7 \; ; \quad m+n \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 7.$$

$$a \equiv 7\left(\frac{m+n}{2}\right) + \frac{11}{2}.$$

$$a \equiv 7K + 30 \pmod{49} (K = 0, 1, 2, 3, 4, 5, 6).$$

$$a \equiv 30, 37, 44, 2, 9, 16, 23 \pmod{49}.$$

Considering both cases we have

$$a = 49n + 2, 3, 9, 16, 23, 30, 37, 44.$$

Transforming to x we get

$$x = 49n + 2, 9, 16, 23, 25, 30, 37, 44.$$

We have thus restricted x to 8 cases modulo 49 instead of the usual 28 cases. The result is that the machine will run longer between solutions.

Linear forms for x were thus calculated for every excluding number $\leqq 67$ and the appropriate links of each chain were supplied with pins. The machine was then set in motion. After running about 2 hours it stopped itself at $x=400453$,

$$a = 2983262201 \; ; \quad a^2 - N = 8889854359815374400 = 2981585880^2 = b^2$$

This gives us $N=(a-b)(a+b)=1676321 \times 5964848081$.

If the entire range had been covered without finding the factors of N, it would have indicated that 3 and 7 were residues contrary to our previous assumption. This case could have been taken care of, without changing the position of the pins, by simply shifting the origin on each chain as explained later. The machine would then have to be set back to zero and another run made. Even if two runs were necessary the consideration of the quadratic character of 3 reduces the number of values of x to 2/9.

Another example of the representation by the difference of two squares is furnished by $N=20191210335106439$, a large factor of $3^{111}+1$. Every factor of the number is of the form $111n+1$. Hence, as before, a can be restricted to one case in $111^2=12321$. Also $a=4n+2$. Consequently we have $a=49284k+6550$. The range for a is $44935627 < a < 10096101675$ corresponding to $W=100{,}000$ as set by congruence tables. Hence

(5) $$a = 49284x + 44978200 \text{ with } 0 < x < 203943 .$$

The linear forms for x were calculated as above with the exception of those for the excluding numbers 3 and 37 which are not prime to the modulus 49284. The first of these was computed as follows. The form (5) may be written

$$a \equiv 36x + 34 (\text{mod } 81), \quad a^2 \equiv 18x + 22 (\text{mod } 81),$$

$$N \equiv 4 (\text{mod } 81), \quad b^2 \equiv 18x + 18 (\text{mod } 81),$$

$$r \equiv 2x + 2 (\text{mod } 9),$$

where r designates all the quadratic residues of 9, which are 0, 1, 4, 7. Putting in these values we obtain

$$x = 9n + 1, 4, 7, 8.$$

The linear forms for 37 were calculated in a similar way. All the linear forms were set on the chains and the machine, after running only 4 seconds, stopped at $x = 145$ giving

$$a = 52124380 \quad b = 26414781.$$

Therefore $N = 20191210335106439 = 25709599 \times 78539161$.

If the same problem had been attacked by the "movable strip" method, the entire range would have been excluded before this solution was discovered.

The machine was also used to show that the other large factor of $3^{111}+1$, namely 64326272436179833 is a prime which gives the complete factorisation

$$3^{111} + 1 = 2^2 \times 7 \times 223 \times 18427 \times 107671 \times 25709599 \times 56737873$$
$$\times 78539161 \times 64326272436179833.$$

The 5th and 7th of these primes are due to Poulet.

We consider next the representation of a number as the sum of two squares. The indeterminate equation $x^2 - 1721y^2 = -1$ has for its fundamental solution

$$x = 31738680901536260 \qquad y = 76506518214341.$$

We have then $x^2 + 1 = 1721y^2$. Thus y is a divisor of the sum of two squares and therefore every factor of y is the sum of two squares. We have

$$y = 17 \times 4500383424373 = 17 \times N.$$

We propose to represent N as the sum of two squares. Using 5 as an excluding number we get $a = 5n + 2, 3$. To save time we consider the two separate cases $a = 5x + 2$ and $a = 5x + 3$. Taking the first case we proceed to exclude values of x with all the excluding numbers except 5. The tables constructed for the difference of squares may be used for the sum of squares. For an excluding number

of the form $4n+1$ the entries are identical. For the case $4n+3$ the entries *not* tabulated are those desired, with the exception of the pair of entries corresponding to $b^2 \equiv 0 \pmod{p}$, which must be included.

The range of a in either case is

$$0 < a < \sqrt{(N/2)}, \quad 0 < a < 1500063, \quad 0 < x < 300013.$$

The machine covered the range for the first case with but a single stop at

$$x = 196113, \quad a = 980567, \quad N - a^2 = 3538871782884.$$

This number is not a square although it is a residue of every prime $\leqq 67$.

To consider the second case, in which $a = 5x+3$, no further calculation was necessary. The values of a in each case must be congruent modulo p. That is

$$5x_1 + 2 \equiv 5x_2 + 3 \pmod{p} \text{ or } x_1 - x_2 = 1/5 \pmod{p}, \text{ for } x_2 \equiv 0, \; x_1 \equiv 1/5 \pmod{p}.$$

On the second start therefore, instead of setting all the chains on their zero positions we set them on the link corresponding to the value of $1/5 \pmod{p}$. This time the machine stopped only once giving

$$x = 157044, \quad a = 785223, \quad b = 1970738. \quad \text{Hence } N = (785223)^2 + (1970738)^2.$$

Since this is a unique representation, it follows that N is a prime. This also gives us the complete factorisation of $(31738680901536260)^2+1$.

By separating the work into two cases, results were obtained in 2/5 of the time required for a single run.

The machine has been used to study the following problem previously considered by Kraitchik,[1] namely to find the least non-square which is a quadratic residue of all primes $\leqq p$.

Following Kraitchik, we consider 0 as a non-residue and the number 8 instead of the prime 2. The numbers sought belong to the forms

$$\left.\begin{matrix} 8n+1 \\ 3n+1 \end{matrix}\right\} \quad 24n+1.$$

Letting $N = 24x+1$ we proceed to exclude values of x with moduli $\geqq 5$. To obtain the necessary linear forms we transform a table of residues by means of the relation $x \equiv (r-1)/24 \pmod{p}$, where r represents the residues of p. All the chains were supplied with pins before starting. To start with, the chain for 5 was provided with a spring and the machine was set in motion. The first non-square solution obtained was 241. Then the next prime was introduced by setting on the spring of the chain for 7. The machine was then run to the first non-square solution after which the spring for the prime 11 was put on,

[1] *Recherches sur la Théorie des Nombres* (Paris, 1924), pp. 41–46.

etc. The squares appearing as solutions are of course squares of primes or products of primes $>p$. These squares were predicted in advance so as to avoid unnecessary reading of the machine. The work was carried up to $p=61$. The results are summarized in the following table, which gives the least non-square N_p that is a residue of all primes $\leqq p$.

$N_3=17$ Prime	$N_{17}=18001$ 47×383	$N_{41}=3206641$ 643×4987
$N_5=73$ Prime	$N_{19}=53881$ Prime	$N_{43}=3818929$ Prime
$N_6=241$ Prime	$N_{23}=87481$ Prime	$N_{47}=9257329$ Prime
$N_7=1009$ Prime	$N_{29}=117049$ 67×1747	$N_{53}=22000801$ Prime
$N_{11}=2641$ 19×139	$N_{31}=515761$ Prime	$N_{59}=48473881$ Prime
$N_{13}=8089$ Prime	$N_{37}=1083289$ Prime	$N_{61}=48473881$ Prime

It is seen that most of the N's are primes. If N_p is a prime it has for residues all the primes $\leqq p$ and also -1. If it is composite there exists some prime $\leqq p$ which is a non-residue of an even number of factors of N. The interesting case of $N_{59}=N_{61}$ is the first of its kind.

The above results confirm and extend the work of Kraitchik. He used the movable strip method with a table having 180 columns and 84 lines to carry the work to N_{47}. A full account of his work is given in his book. A few errors are to be noted which however do not happen to influence the final results. It should be pointed out that to extend Kraitchik's work by his method would imply the construction of an entirely new table the size of which must be determined in advance. Using the machine the work may be extended indefinitely and is complete as far as it goes.

ERRORS IN KRAITCHIK'S TABLE III.

Théorie des Nombres, pp. 190–9

p	N	for	read
31	5	$a=8$	$a=11$
31	5	$t=11$	$t=8$
47	6	$t=10$	$t=11$
47	34	$t=19$	$t=22$

Since writing the above, the author's attention has been directed to Kraitchick's new second volume (Théorie des Nombres, vol. 2, Paris, 1926) which contains tables of $a \pmod p$ for $p\leqq67$. The above errors have been corrected. Only two new errors appear, namely:

p	N	for	read
59	23	$x=14$	$x=15$
59	44	$x=14$	$x=17$

STRONG AND WEAK INEQUALITIES INVOLVING THE RATIO OF TWO CHORDS OR TWO ARCS OF A CIRCLE; CHAINS OF INEQUALITIES

By HARVEY A. SIMMONS, Northwestern University

1. Introduction. It is known that if two arcs OA and OB, with arc $OA <$ arc $OB \leqq \pi a$ be taken on a circle of radius a (c.f. Fig. 1), then[1]

$$OA/OB > \text{arc } OA/\text{arc } OB \tag{1}$$

Furthermore it is easy to prove that $(OA/OB)^2$ does not exceed arc OA/arc OB for all chords OA, OB, with $OA < OB$, that can be taken on a semi-circumference.[2] Owing to this fact, the inequality (1) might be called *strong*. In this paper an inequality $p^n > q$ will be called *strong* if it is true for all pairs of values of p, q on the interval in which they are considered while the inequality $p^{n+1} > q$ is not true for some choice of p, q on this interval, n being a positive integer.

One purpose of this paper is to establish several *strong* inequalities, each of which involves either the ratio of two chords of a circle, or that of two arcs, and to obtain other inequalitites which it seems fitting to call *weak*, from the *strong* ones.[3] Another purpose is to develop two interesting chains of inequalities one of which contains as its last link the inequality that Professor Uspensky stated in Theorem III of his article referred to above (c.f. Theorem VII below). Incidentally, it seems possible that the numerous new trigonometric inequalities in this paper may stimulate some reader to make another geometric proof by "curious" induction.

When unspecified, the units of length, area, and volume in this paper will be understood to be arbitrary.

Let a semi-circle in the xy-plane (see Fig. 1) be represented by the equation

$$r = 2a \sin\theta, \quad 0 \leqq \theta \leqq \pi/2, \tag{2}$$

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[1] A geometric proof of (1) was given by J. V. Uspensky, this Monthly, vol 34, May (1927), p. 249.

[2] An example in which $(OA/OB)^2 <$ arc OA/arc OB is afforded by taking θ in equation (2) to be 30° for OA and 90° for OB. For $(a^2/4a^2) < [(\pi a \div 3)/\pi a]$. An example in which $(OA/OB)^2 >$ arc OA/arc OB is afforded by taking θ in equation (2) to be 60° for OA and 90° for OB. For $(3a^2/4a^2) > [(2\pi a \div 3)/\pi a]$.

[3] These terms "strong" and "weak" are not to be understood in a sense which is often used in connection with inequalities, namely that $z > y$ is a stronger inequality than $x > y$ if $z > x$. Our meaning is in accord with the usage by which a first theorem is said to be stronger than a second theorem if the conclusions in both theorems are the same, while the hypotheses of the first are true for a larger class than that for which the hypotheses of the second hold. For example, of the two theorems (I) if $x > 1$, $x > 0$ and (II) if $x > 2$, $x > 0$ the former is the stronger. In this paper strong inequalities are those that lead to, or express strong theorems.

and denote the coördinates of A and B, two points on the circumference, by (r_1, θ_1) and (r_2, θ_2), respectively, where $0<\theta_1<\theta_2\leqq\pi/2$, and denote by a_1, a_2 the lengths of the arcs OA, OB, respectively. Let k_1 and k_2 denote the areas of the segments OmA and OmB, respectively, of the circle. Let v_1 and v_2 denote the volumes of the segments of the sphere (obtained by rotating the semi-circle about the y-axis) which lie below the planes ZOA and ZOB, respectively, OZ being perpendicular to the xy-plane. Let s_1 and s_2 denote the areas of the spherical portions of the former and latter segments, respectively. Using these definitions we can prove the following six theorems.[1]

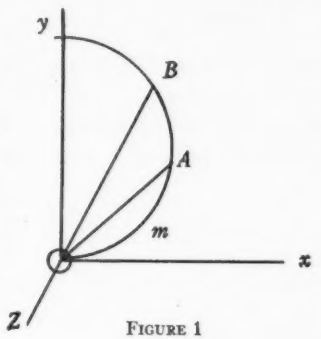

FIGURE 1

THEOREM I. *The chords r_1, r_2 and the areas k_1, k_2 satisfy the inequality $(r_1/r_2)^3>k_1/k_2$, which is strong.*

THEOREM II. *The arcs a_1, a_2 and the areas k_1, k_2 satisfy the inequality $(a_1/a_2)^2>k_1/k_2$, which is strong.*

THEOREM III. *The chords r_1, r_2 and the areas s_1, s_2 satisfy the inequality $(r_1/r_2)^2>s_1/s_2$, which is strong.*

THEOREM IV. *The arcs a_1, a_2 and the areas s_1, s_2 satisfy the inequality $a_1/a_2>s_1/s_2$, which is strong.*

THEOREM V. *The chords r_1, r_2 and the volumes v_1, v_2 satisfy the inequality $(r_1/r_2)^4>v_1/v_2$, which is strong.*

THEOREM VI. *The arcs a_1, a_2 and the volumes v_1, v_2 satisfy the inequality $(a_1/a_2)^2>v_1/v_2$, which is strong.*

2. Proofs of Theorem I and Theorem II. To prove Theorem I, we first observe that $(r_1/r_2)^3=\sin^3\theta_1/\sin^3\theta_2$, by (2), and that the areas k_1, k_2 have the values

$$k_1=\frac{1}{2}\int_0^{\theta_1}4a^2\sin^2\theta\,d\theta=a^2(\theta_1-\sin\theta_1\cos\theta_1),\quad k_2=a^2(\theta_2-\sin\theta_2\cos\theta_2).$$

[1] In this paper, we prove only the first two theorems, which are the ones that we use in §4. Our method of proof is essentially the same for all six of the theorems and a good deal of repetition is avoided by the omission.

We need to prove, then, that

$$(3)\qquad \frac{\sin^3\theta_1}{\sin^3\theta_2} > \frac{\theta_1 - \sin\theta_1\cos\theta_1}{\theta_2 - \sin\theta_2\cos\theta_2}, \quad \text{or} \quad \frac{\sin^3\theta_1}{\theta_1 - \sin\theta_1\cos\theta_1} > \frac{\sin^3\theta_2}{\theta_2 - \sin\theta_2\cos\theta_2}.$$

Since $0<\theta_1<\theta_2\leqq\pi/2$, the inequality (3) will be established if it is shown that $y=\sin^3x/(x-\sin x\cos x)$ decreases as x increases from 0 to $\pi/2$, the value $x=0$ itself being excluded. Differentiating y with respect to x and simplifying the result by means of the identity $\sin^2x+\cos^2x=1$, we find

$$y' = \frac{\sin^2 x(3x\cos x - 3\sin x + \sin^3 x)}{(x - \sin x\cos x)^2}.$$

In this equation, $y'<0$ if $3x\cos x-3\sin x+\sin^3x<0$, or if

$$(4)\qquad x < \tfrac{2}{3}\tan x + \tfrac{1}{3}\sin x\cos x,\ 0 < x \leqq \pi/2.$$

To prove (4), consider the curves

$$y = x,\ y = \tfrac{2}{3}\tan x + \tfrac{1}{3}\sin x\cos x.$$

Each of these curves passes through the origin with slope 1, but the latter curve has at every point the slope $\frac{2}{3}\sec^2x+\frac{1}{3}\cos2x$, which is greater than 1 throughout the interval $0<x\leqq\pi/2$; for

$$\frac{2\sec^2 x}{3} + \frac{\cos 2x}{3} \equiv \frac{2 + 2\cos^4 x - \cos^2 x}{3\cos^2 x} > 1$$

since $(\cos^2x-1)^2>0$ on the interval $0<x\leqq\pi/2$. Consequently, on this interval the latter curve lies above the former. Hence (4) holds, $y'<0$, and the function $\sin^3x/(x-\sin x\cos x)$ decreases, as was to be proved. This proves $(r_1/r_2)^3 > k_1/k_2$. That this inequality is strong, and hence that Theorem I is true, now follows from the example below in which $(r_1/r_2)^4<k_1/k_2$.

Example. Take $\theta_1=30°$ and $\theta_2=60°$ in (2). Then $(r_1/r_2)^4<k_1/k_2$ since

$$\frac{\sin^4 30°}{\sin^4 60°} < \frac{(\pi/6) - \sin 30°\cos 30°}{(\pi/3) - \sin 60°\cos 60°}, \qquad [c.f.\ (3)].$$

Before proving Theorem II, we pause to give a geometric interpretation of inequality (4). Let C (see Fig. 2) be the center of a circle of radius $CA=1$, AQ a tangent to the circle, P the point where CQ intersects the circumference, M the foot of the perpendicular dropped from P upon CA, and the angle $ACP=x$, where $0<x\leqq\pi/2$. Then the inequality

$$(3x)/2 < \tan x + \tfrac{1}{2}\sin x\cos x,$$

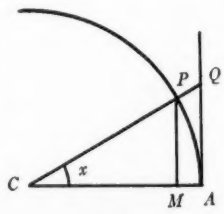

FIGURE 2

which is equivalent to (4), reads: (three times the area of the sector CAP) is less than (twice the area of the triangle CAQ) plus (the area of the triangle CMP).

To prove Theorem II, we first observe that

$$\frac{a_1^2}{a_2^2} = \frac{\theta_1^2}{\theta_2^2}, \quad \frac{k_1}{k_2} = \frac{\theta_1 - \sin\theta_1\cos\theta_1}{\theta_2 - \sin\theta_2\cos\theta_2}.$$

We need to prove, then, that

$$(5) \qquad \frac{\theta_1^2}{\theta_2^2} > \frac{\theta_1 - \sin\theta_1\cos\theta_1}{\theta_2 - \sin\theta_2\cos\theta_2}, \text{ or } \frac{\theta_1^2}{\theta_1 - \sin\theta_1\cos\theta_1} > \frac{\theta_2^2}{\theta_2 - \sin\theta_2\cos\theta_2}.$$

To establish (5), we need only to show that $y = x^2/(x - \sin x\cos x)$ decreases as x increases from 0 to $\pi/2$, the value $x=0$ itself being excluded, since $0<\theta_1<\theta_2\leqq\pi/2$. Differentiating y with respect to x and simplifying the result by means of the identity $\sin^2 x+\cos^2 x=1$, we find

$$y' = \frac{2x\cos x\,(x\cos x - \sin x)}{(x - \sin x\cos x)^2} < 0,$$

since[1] $x<\tan x$ on the interval $0<x\leqq\pi/2$. Hence (5) is true. This proves that $(a_1/a_2)^3 > k_1/k_2$. That this inequality is strong, and hence that Theorem II is true, now follows from the

Example. Take $\theta_1=30°$ and $\theta_2=60°$. Then $(a_1/a_2)^3<k_1/k_2$ since

$$\frac{(\pi/4)^3}{(\pi/3)^3} < \frac{(\pi/6) - \sin 30°\cos 30°}{(\pi/3) - \sin 60°\cos 60°}, \qquad \text{[see (5)]}.$$

3. Corollaries to the above theorems. By definition $r_1<r_2$ (see §1), so that $r_1/r_2<1$. Hence if $(r_1/r_2)^n>N$, where N is a positive number and n is an integer greater than 1, the powers $1, \cdots, n-1$ of r_1/r_2 exceed N. These statements about r_1/r_2 apply also to a_1/a_2. Hence inequalities, which it seems proper to call *weak*, follow from five of the above theorems. These inequalities are expressed in the following corollaries.

COROLLARY I. $(r_1/r_2)^2>k_1/k_2$, $r_1/r_2>k_1/k_2$, from Theorem I.

COROLLARY II. $a_1/a_2>k_1/k_2$, from Theorem II.

COROLLARY III. $r_1/r_2>s_1/s_2$, from Theorem III.

COROLLARY IV. $(r_1/r_2)^3>v_1/v_2$; $(r_1/r_2)^2>v_1/v_2$; $(r_1/r_2)>v_1/v_2$; from Theorem V.

COROLLARY V. $a_1/a_2>v_1/v_2$, from Theorem VI.

[1] This known inequality may be seen from the fact that the area (in Fig. 2) of the sector CAP is less than the area of the triangle CAQ.

4. Chains of inequalities. In this section we apply the first two theorems above to obtain two chains of inequalities which it seems proper to include here. The proof of the third inequality in the first chain [see (6)] is the most difficult part of this paper, the difficulty coming from the fact that $\sin^2 x/x$ increases on part of the interval from 0 to $\pi/2$ and decreases on the rest of it.

Let P and Q be two polygons inscribed in the same circle of radius a. Denote by $p_1, p_2, \cdots, p_n$ the lengths of the sides of P in increasing order of magnitude and by $u_1, u_2, \cdots, u_n$ their respective intercepted arcs (each $\leqq \pi a$) on the circumference, which will be taken as the unit of arc. Denote by $q_1, q_2, \cdots, q_m$ the lengths of the sides of Q in decreasing order of magnitude, and by $v_1, v_2, \cdots, v_m$ the lengths of their respective intercepted arcs, and suppose $q_1 < p_1$. Denote by j_i $(i=1, 2, \cdots, n)$ the area of the segment of the circle which is bounded by the i-th side of P and its intercepted arc, and by k_t $(t=1, 2, \cdots, m)$ the area which is bounded by the t-th side of Q and its intercepted arc. Let $J=j_1+j_2+\cdots+j_n$ and $K=k_1+k_2+\cdots+k_m$. Using these definitions, we shall prove

THEOREM VII. *The numbers* p_i, u_i, j_i $(i=1, 2, \cdots, n)$, q_t, v_t, k_t $(t=1, 2, \cdots, m)$, J, K *satisfy the inequalities*

$$(6)\quad \frac{J}{K}=\frac{j_1+j_2+\cdots+j_n}{k_1+k_2+\cdots+k_m}>\frac{p_1^3+p_2^3+\cdots+p_n^3}{q_1^3+q_2^3+\cdots+q_m^3}>\frac{p_1^2+p_2^2+\cdots+p_n^2}{q_1^2+q_2^2+\cdots+q_m^2}$$
$$>1>\frac{p_1+p_2+\cdots+p_n}{q_1+q_2+\cdots+q_m},$$

$$(7)\quad \frac{J}{K}>\frac{u_1^2+u_2^2+\cdots+u_n^2}{v_1^2+v_2^2+\cdots+v_m^2}>\frac{u_1+u_2+\cdots+u_n}{v_1+v_2+\cdots+v_m}=1.$$

PROOF OF (6). The equality sign holds by definition. To prove the first inequality, we recall that $p_1 \leqq p_2 \leqq \cdots \leqq p_n$ and $q_1 \geqq q_2 \geqq \cdots \geqq q_m$, by definition, so that $j_1 \leqq j_2 \leqq \cdots \leqq j_n$ and $k_1 \geqq k_2 \geqq \cdots \geqq k_m$. Therefore, by Theorem I,

$$\frac{p_1^3}{j_1} \geqq \frac{p_2^3}{j_2} \geqq \cdots \geqq \frac{p_n^3}{j_n} \text{ and } \frac{q_1^3}{k_1} \leqq \frac{q_2^3}{k_2} \leqq \cdots \leqq \frac{q_m^3}{k_m}.$$

Consequently

$$\frac{p_1^2}{j_1} \geqq \frac{p_1^3+p_2^3+\cdots+p_n^3}{j_1+j_2+\cdots+j_n} \text{ and } \frac{q_1^3}{k_1} \leqq \frac{q_1^3+q_2^3+\cdots+q_m^3}{k_1+k_2+\cdots+k_m}.$$

Now $q_1 < p_1$, by definition, and it follows from Theorem I that $(q_1^3/k_1) > (p_1^3/j_1)$. Hence

$$\frac{q_1^3 + q_2^3 + \cdots + q_m^3}{k_1 + k_2 + \cdots + k_m} > \frac{p_1^3 + p_2^3 + \cdots + p_n^3}{j_1 + j_2 + \cdots + j_n},$$

which is equivalent to the first inequality of (6).

The second inequality of (6) follows from the facts

$$p_1^3 + p_2^3 + \cdots + p_n^3 \geqq p_1(p_1^2 + p_2^2 + \cdots + p_n^2),$$
$$q_1^3 + q_2^3 + \cdots + q_m^3 \leqq q_1(q_1^2 + q_2^2 + \cdots + q_m^2),$$

so that

$$\frac{p_1^3 + p_2^3 + \cdots + p_n^3}{q_1^3 + q_2^3 + \cdots + q_m^3} \geqq \frac{p_1(p_1^2 + p_2^2 + \cdots + p_n^2)}{q_1(q_1^2 + q_2^2 + \cdots + q_m^2)} > \frac{p_1^2 + p_2^2 + \cdots + p_n^2}{q_1^2 + q_2^2 + \cdots + q_m^2},$$

the last inequality holding because $q_1 < p_1$ by definition.

The third inequality of (6) is equivalent to

$$\frac{p_1^2 + p_2^2 + \cdots + p_n^2}{u_1 + u_2 + \cdots + u_n} > \frac{q_1^2 + q_2^2 + \cdots + q_m^2}{v_1 + v_2 + \cdots + v_m},$$

since $u_1 + u_2 + \cdots + u_n = v_1 + v_2 + \cdots + v_m = 1$ by our choice of the unit of arc; and this inequality, in turn, is equivalent to the following one which we desire to prove

$$(8) \qquad \frac{\sin^2\theta_1 + \sin^2\theta_2 + \cdots + \sin^2\theta_n}{\theta_1 + \theta_2 + \cdots + \theta_n} > \frac{\sin^2\phi_1 + \sin^2\phi_2 + \cdots + \sin^2\phi_m}{\phi_1 + \phi_2 + \cdots + \phi_m},$$

where the $\theta_i (i = 1, 2, \cdots, n)$ and $\phi_t (t = 1, 2, \cdots, m)$ are the angles at the center of the circle which intercept on the circumference arcs of lengths $u_i/2$ and $v_t/2$ respectively. The definitions of the u_i, v_t imply that

$$(9) \qquad \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_n \leqq \pi/2,\ \phi_1 \geqq \phi_2 \geqq \cdots \geqq \phi_m > 0,\ \phi_1 < \theta_1,$$

$$\theta_1 + \theta_2 + \cdots + \theta_n = \phi_1 + \phi_2 + \cdots + \phi_m = \pi,\ m > n \geqq 3.$$

Now since ϕ_1 is the largest ϕ_t, (8) will follow if we prove that

$$(10) \qquad (\sin^2\theta_i)/\theta_i > (\sin^2\phi_1)/\phi_1$$

for every i. To do this, we need the two properties (P_1) and (P_2), stated below, of the function $y = \sin^2 x/x$; these properties of y following from an interpretation of it derivative

$$y' = [\sin x(2x\cos x - \sin x)]/x^2.$$

(P_1). On the interval $0 < x < X$, where X is the smallest positive solution of the equation $2x\cos x - \sin x = 0$ and lies between 66° and 67°,[1] y increases with x. When $x = \pi/4$, $y = 2/\pi$ and when $x = X$, $y = \sin^2 X/X$.

[1] If $x = 66°$, $2x = 2.3038 > \tan x = 2.2460$; if $x = 67°$, $2x = 2.3387 < \tan x = 2.3559$.

(P_2). On the interval $X<x\leqq\pi/2$, y decreases as x increases. When $x=\pi/2$, $y=2/\pi$.

Since $n\geqq 3$, by (9), $\theta_1\leqq 60°<X$ and consequently $\phi_1<60°$. Hence, by (P_1) and (9), (10) holds unless $\theta_i>X(>66°)$; and since $\theta_1+\theta_2+\cdots+\theta_n=\pi$, by (9), there are at most two values of i for which $\theta_i>X$. Hence we only need to prove (10) in the following cases: (a) when $\theta_{n-1}\leqq X<\theta_n$; (b) when $\theta_{n-2}<X<\theta_{n-1}\leqq\theta_n$.

(a). If $n\geqq 4$, $\theta_1<[(\pi-X)/3]<38°$, and consequently $\phi_1<38°$ by (9). Then if $i<n$, (10) holds, by (P_1). When $i=n$, the left hand side of (10) has a value $\geqq 2/\pi$, by (P_2), while the right hand side is less than $2/\pi$, by (P_1). Hence (10) holds when $n\geqq 4$. The case $n=3$ we argue as follows. If $i<3$, (10) holds, by (P_1). If $i=3$, let $\theta_3=\frac{1}{2}\pi-z$. As $\theta_3>X$, it follows that

$$\theta_1\leqq\frac{\pi-(\frac{1}{2}\pi-z)}{2}=\frac{\pi}{4}+\frac{z}{2},\quad 0<z<\left(\frac{1}{2}\pi-X\right).$$

so that $\phi_1<\frac{1}{4}\pi+\frac{1}{2}z$ on the specified interval. By (P_2), the ratio

$$\sin^2(\tfrac{1}{2}\pi-z)/(\tfrac{1}{2}\pi-z)$$

increases with z on its interval. Hence we shall prove (10) if we show that

$$\frac{\sin^2(\frac{1}{4}\pi+\frac{1}{2}z)}{\frac{1}{4}\pi+\frac{1}{2}z}<\frac{\sin^2(\frac{1}{2}\pi-z)}{\frac{1}{2}\pi-z},\quad 0<z<\left(\frac{1}{2}\pi-X\right).\tag{11}$$

But (11) is equivalent to

$$z>\frac{\pi\sin z}{2(2-\sin z)},\quad 0<z<\left(\frac{1}{2}\pi-X\right),$$

which follows from the fact that on the specified interval $\sin z<z$. We leave the proof to the reader. Hence (10) holds in case (a).

(b). The method used in (a) to handle the case $n\geqq 4$ also applies here. If $n=3$, since θ_2 and θ_3 both exceed $X(>66°)$, $\theta_1=\pi-\theta_2-\theta_3<48°$, and consequently $\phi_1<48°$. Hence if $i=1$, (10) holds by (P_1). By (P_2) and the definitions of θ_2 and θ_3, (10) holds for $i=2$ if it holds for $i=3$. If $\phi_1<\pi/4$, $(\sin^2\phi_1/\phi_1)<2/\pi$ by (P_1), and $2/\pi\leqq\sin^2\theta_3/\theta_3$, by P_2; therefore (10) holds. Hence we only need to consider the case $\pi/4<\phi_1<48°$. Then $\theta_3<69°$, by (9) and the inequalities $66°<X<\theta_2<\theta_3$. To obtain the desired result, we put $z=6°$ in (11), and find

$$\frac{\sin^2(4\pi/15)}{4\pi/15}<\frac{\sin^2(7\pi/15)}{7\pi/15}.$$

The left member of this inequality exceeds any value that $\sin^2\phi_1/\phi_1$ can have on the interval $\pi/4<\phi_1<48°$, by (P_1); and the right member is less than any

value that $\sin^2 \theta_3/\theta_3$ can have on the interval $X<\theta_3<69°$, by (P_2). Hence (10) holds for every i.

One can now prove (8), and thus complete the proof of the third inequality in (6), by arranging the ratios $\sin^2 \theta_i/\theta_i$ $(i=1, 2, \cdots, n)$ in increasing order of magnitude and using the method which was employed above in proving the first inequality of (6).

The fourth, and last, inequality of (6) can be proved by use of inequality (1) and the method employed above in proving the first inequality of (6), as was shown by Professor Uspensky[1]. Hence (6) is true.

PROOF OF (7). As our proof of (7) is not difficult, we merely describe it. The first inequality follows from Theorem II in the same way that the first inequality of (6) followed from Theorem I; u, v, and the exponent 2 playing the rôle here which p, q, and the exponent 3, respectively, played there. The method which was used in proving the second inequality of (6) can be employed to prove the second inequality of (7). The equality sign in (7) holds because $u_1+u_2+\cdots+u_n=v_1+v_2+\cdots+v_m$. Hence Theorem VII follows.

The corollaries below can be obtained by a small amount of inference from (6), (7), and the definitions of the symbols involved.

COROLLARY VI. If $\theta_i(i=1, 2, \cdots, n)$ and $\phi_t(t=1, 2, \cdots, m)$ are positive angles, each $\leqq \pi/2$, such that $\theta_1+\theta_2+\cdots+\theta_n=\phi_1+\phi_2+\cdots+\phi_m=\pi$, and if the smallest θ_i is greater than the largest ϕ_t, and if λ is a positive integer, the following inequalities hold:

$$\frac{\sum_{i=1}^{n}(\sin\theta_i)^{\lambda+1}}{\sum_{t=1}^{m}(\sin\phi_t)^{\lambda+1}} > \frac{\sum_{i=1}^{n}(\sin\theta_i)^{\lambda}}{\sum_{t=1}^{m}(\sin\phi_t)^{\lambda}}; \quad \frac{\sum_{i=1}^{n}(\theta_i)^{\lambda+1}}{\sum_{t=1}^{m}(\phi_t)^{\lambda+1}} > \frac{\sum_{i=1}^{n}(\theta_i)^{\lambda}}{\sum_{t=1}^{m}(\phi_t)^{\lambda}} \geqq 1.$$

COROLLARY VII. If the polygons P, Q, described above, are regular and have sides of lengths p, q, respectively, and if J, K are as defined above, the following inequalities hold:

$$\frac{J}{K} > \frac{np^3}{mq^3} > \frac{np^2}{mq^2} > 1 > \frac{np}{mq}.$$

If m, or n and m, are sufficiently large, the first inequality of Corollary VII gives results which no existing tables of logarithms are accurate enough to detect. We give an example in which four-place tables are inadequate.

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[1] Loc. cit., p. 250.

Example. Suppose the regular polygons P, Q have 60, 120 sides, respectively, and that they are inscribed in the same circle. Then the first inequality of Corollary VII says

$$J/K > (\sin^3 3°)/(2 \sin^3 1°30').$$

COROLLARY VIII. Denote by A and A' the areas of two polygons P and Q, respectively, which are inscribed in the same circle. If the shortest side of P is greater than the longest side of Q, then $A < A'$.

Remark. The type of reasoning used in proving the third inequality of (6) can be used to prove directly that $[(p_1^3+p_2^3+\cdots+p_n^3)/(q_1^3+q_2^3+\cdots+q_m^3)]>1$, the function $\sin^3 x/x$ playing the rôle here which $\sin^2 x/x$ played there. To prove directly that $(np^3/mq^3)>1$, (see Corollary VII), we can replace this inequality by the equivalent one: $n \sin^3(\pi/n) > m \sin^3(\pi/m)$, $m>n\geqq 3$, and then obtain the desired result by considering the function $y=x \sin^3(\pi/x)$, where x has a continuous range over all real numbers $\geqq 3$.

In all three of the proofs just mentioned, the case $n=3$ requires special consideration, while the other cases, $n=4, 5, \cdots$, ad infinitum, naturally call for simultaneous treatment. We are inclined to call this procedure an analog in analysis to Professor Uspensky's geometric induction.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. THE LOWEST COMMON MULTIPLE OF n POLYNOMIALS

By ARNOLD DRESDEN, Swarthmore College

1. It is shown in elementary algebra that the product of two polynomials is equal to the product of their highest common factor (H.C.F.) and their lowest common multiple (L.C.M.). Since the H.C.F. of two polynomials can always be found by means of the Euclidean algorithm, this theorem furnishes a direct method for determining the L.C.M. of two polynomials. For the case of three or more polynomials, the determination of the H.C.F. and L.C.M. could be carried out by a step by step process, on the basis of the recursion formulas developed in §2. While this is not extensively laborious for the H.C.F. it involves more work than is necessary for the L.C.M. The purpose of the

present note is to generalize the formula for the case of two polynomials referred to above and thus to obtain another method for finding the L.C.M. of n polynomials.

2. We shall denote the H.C.F. and the L.C.M. of two polynomials A_1 and A_2 by $H(A_1, A_2)$ and $L(A_1, A_2)$ respectively, or, briefly, by H_{12} and L_{12}. The H.C.F. and the L.C.M. of n polynomials $A_1, \cdots, A_n$ will be denoted by $H_{12\ldots n}$ and $L_{12\ldots n}$ respectively.

It is readily seen that $H_{12\ldots n}=H(H_{12\ldots,n-1}, A_n)$. For if $A_k=H_{12\ldots,n-1}B_k$, $k=1, \cdots, n-1$, then $B_1, B_2, \cdots, B_k$ have no factor in common. If, moreover, $H_{12\ldots,n-1}=\alpha H$, and $A_n=\alpha A$, where A and H are relatively prime, then $H(H_{12\ldots,n-1}, A_n)=\alpha$. Furthermore $A_k=\alpha HB_k$; since A and HB_k are without common factor (because none of the factors of A occur in H, and if a factor of A were present in every B_k these polynomials would have a factor in common), it follows that $H_{12\ldots n}$ is also equal to α.

To show that a similar relation holds for the L.C.M., we observe that a necessary and sufficient condition that L is the L.C.M. of the polynomials $A_1, A_2, \cdots, A_n$ is that there exist a set of polynomials $C_1, C_2, \cdots, C_n$, without common factor, such that $L=A_kC_k$, $k=1, \cdots, n$, (the polynomials C_k are then necessarily unique). For, if these polynomials exist, L is clearly a common multiple of $A_1 \cdots, A_n$; and if L_1 is an arbitrary common multiple of $A_1, \cdots, A_n$, there must exist polynomials $D_1, \cdots, D_n$ such that $L_1=A_kD_k$ and hence such that $LD_k=L_1C_k$, $k=1, \cdots, n$. If there were then any factor of L which was not a factor of L_1, it would have to be a factor of every C_k, so that these polynomials would not be relatively prime; therefore every multiple of $A_1, \cdots, A_n$ is divisible by L, so that L is indeed the L.C M. of these polynomials.

We see now readily that $L_{12\ldots n}=L(L_{12\ldots,n-1}, A_n)$. For there exist polynomials $B_1, \cdots, B_{n-1}$ without common factor, such that $A_kB_k=L_{12\ldots,n-1}$, $k=1, \cdots, n-1$. Suppose furthermore that $L_{12\ldots,n-1}=\alpha L$ and that $A_n=\alpha A$, where A and L are relatively prime. Then $A_nL=A_kB_kA=\alpha AL$, $k=1, \cdots, n-1$ and the polynomials $L, B_1A, \cdots, B_{n-1}A$ are without common factors, because none of the factors of L occurs in A, and none of these factors can be present in every B_k, $k=1, \cdots, n-1$, which were supposed to be relatively prime. Hence $L(L_{12\ldots,n-1}, A_n)=\alpha AL=L_{12\ldots,n}$.

3. We note the obvious fact, that for every polynomial p, $H(pA, pB)=pH(A, B)$.

We define a set of polynomials $H_1, \cdots, H_n$ as follows:

$$H_1 = H, \quad H_2 = H_{12}, \quad H_k = H(A_1A_2 \cdots A_{k-1}, \quad H_{k-1}A_k), \quad \text{for } k = 3, \cdots, n.$$

The desired formula is then obtained as follows:

$$L_{123} = L(L_{12}, A_3) = \frac{L_{12}A_3}{H(L_{12}, A_3)} = \frac{A_1A_2A_3/H_2}{H(A_1A_2/H_2, A_3)}$$

$$= \frac{A_1A_2A_3}{H(A_1A_2, H_2A_3)}\ ;\ \text{i.e. } A_1A_2A_3 = L_{123}H_3.$$

Suppose now that $A_1A_2 \ldots A_{n-1} = L_{12\ldots,n-1} \cdot H_{n-1}$. Then

$$L_{12\ldots n} = L(L_{12\ldots,n-1},\ A_n) = \frac{L_{12\ldots,n-1}A_n}{H(L_{12\ldots,n-1}, A_n)} = \frac{A_1A_2\ldots A_n}{H_{n-1}\cdot H(A_1\ldots A_{n-1}/H_{n-1},\ A_n)}$$

$$= \frac{A_1A_2\ldots A_n}{H(A_1A_2\ldots A_{n-1},\ A_nH_{n-1})}\ ;\ \text{i.e. } A_1A_2 \cdots A_n = L_{12\ldots n}\cdot H_n.$$

This formula, of which the validity has thus been established, is the generalization of the familiar result of elementary algebra referred to in the opening paragraph; it furnishes a simple method for the calculation of the L.C.M. of n polynomials.

II. Note on the Calculation of Standard Deviations

By Vladimir Rojansky, University of Minnesota

The object of this note is to describe a simple device which can be effectively used to shorten the work of systematic calculation of standard deviations of sets of numbers. The practicability of the device depends on the range of the sets; the smaller the range, the handier the device. The limitations due to range will be apparent after the method has been described.

The standard deviation of a set of n numbers $x_1, x_2, \cdots, x_n$, is ordinarily computed by direct substitution into the formula:

$$\sigma = \left[\frac{1}{n}\ \sum_{i=1}^{n}(x_i - M_a)^2 - (M - M_a)^2\right]^{1/2},$$

where M is the arithmetic mean of the set, and M_a is a conveniently chosen "assumed mean." The cumbersome part of the calculation is the evaluation of the first term within the brackets.

The present method can best be described by an illustration, which, to save space, will be extremely simple. Let it be desired, for example, to calculate the standard deviations of a series of sets of integers, the integers in each set ranging, say from 350 to 360. Two strips of paper are then prepared. On one, which will be referred to as A, the consecutive integers from 350 to 360 are written in a vertical column. On the second, which will be referred to as B, a column of squares of consecutive integers, say, from -5 to $+5$ is written. The alignment of the entries on each strip is uniform, and is the same on both strips.

Let a particular set of numbers whose standard deviation is to be found consist of the following entries: 360, 351, 360, 352 and 360. The procedure is then as follows:

A strip of paper, which will be referred to as W, is placed beside A, and check marks are made on it opposite the numbers on A which correspond to those read from the records. The numbers on A, marked W, are added on an adding machine, care being taken of the frequencies. Division by the number of entries gives the mean, which is written out on W. An assumed mean is chosen and recorded on W, and the corresponding number on A is conspicuously marked on W. The work in this particular example appears as follows:

$$M = 1783 \div 5 = 356.6\,; \quad M_a = 356\,; \quad M - M_a = .6.$$

A	W	B	W
350		25	—
351	—	16	—
352	—	9	
353		4	
354		1	
355		0	*
356	*	1	
357		4	
358		9	
359		16	—||
360	—||	25	
1783	5	89	5

Strip W is then removed from A and placed beside B in such a way that the mark of the assumed mean on W falls opposite the zero on B. The numbers on B, checked on W, are added, care being taken of the frequencies, and thus the sum of the squares of the deviations of the entries from the assumed mean is obtained. The work appears as shown.

The standard deviation is then found to be:

$$\sigma = [89 \div 5 - (.6)^2]^{1/2} = 4.176.$$

The example just given describes the essence of the method, the chief advantages of which are: it practically eliminates tabulation of intermediate steps, and makes it unnecessary actually to evaluate the deviations of the entries from the assumed mean, and their squares. For these reasons it makes the work less tedious and much more rapid than does the direct substitution into the formula. The main disadvantages of the method are: it leaves no comprehensible record of the steps taken, and for large ranges the strips become long and difficult to handle, while using smaller figures to shorten the strips increases the possibility of misplaced check-marks.

Particular conditions are, of course, to determine the relative importance of the merits and the disadvantages. With respect to the last objection it may be said, however, that, first, it does not apply in a good part of the cases arising in the use of statistical methods in natural sciences, where the ranges are usually not great, and, secondly, it may be partly obviated by placing the columns A and B on two thin circular drums mounted on a common axle, and by putting the frequency marks on a third drum, placed between the first two. (This drum may be a part of drum A.) Careful construction may then

extend the applicability of the device to ranges as great as from 1 to 1000 without prohibitive bulkiness. Several simplifications may also be used in particular cases. For instance, if it is of no advantage to calculate the means from column A it becomes unnecessary to write out the numbers on A in full, so that the flexibility of the device increases.

The method was devised while the writer was temporarily employed at the Institute of Anatomy of the University of Minnesota, and it is a pleasure for him to acknowledge the encouragement and helpful advice he received from Dr. Richard E. Scammon.

III. A Corollary to Cauchy's Integral Formula

By Norman Miller, Queens University

A well known theorem in the theory of functions of a complex variable states that if $f(z)$ is continuous on a curve C open or closed, then the integral $(1/2\pi i)\int_C f(t)dt/(t-z)$ denotes an analytic function of z in any connected region of the plane which does not include points of C. If, in particular, C is a closed curve and f is analytic in the closed region bounded by C then the integral represents the function $f(z)$ at all points within C and the function zero outside C. The purpose of the present note is to prove the following extension of this theorem.

Theorem: *Suppose that $f(z)$ is a function which in a certain region R is single valued and analytic wherever defined and has no removable singularities; suppose also that C is a closed curve within R, on which $f(z)$ is analytic and within which it has at most a finite number of isolated singular points; if $\phi(z)$ is defined within C by the formula $(1/2\pi i)\int_C f(t)dt(t-z)$ and defined on and outside C by analytic continuation of this function wherever possible, and if $\psi(z)$ is defined outside C by the same formula and defined on and within C by analytic continuation wherever possible, then $f(z)$ is identical with $\phi(z)-\psi(z)$ at all points of R where f is defined.*

Let $f(z)$ have, within C, k singular points having the principal parts

$$P_1(z), P_2(z), \cdots, P_k(z) \text{ and let } P(z) \equiv P_1(z)+P_2(z)+\cdots+P_k(z).$$

Then $f(z)=\bar{f}(z)+P(z)$ where $\bar{f}(z)$ has only removable singularities within C, which we shall suppose removed. If z is a point within C we have

$$\phi(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z} = \bar{f}(z) + \frac{1}{2\pi i}\int_C \frac{P(t)dt}{t-z},$$

of which the latter term is the sum of k terms of the form $(1/2\pi i)\int_C P_i(t)dt/(t-z)$.

If $P_i(z)$ is the principal part of a pole at a point z_0 it consists of a finite number of terms of the form $a_n/(z-z_0)^n$, n being a positive integer. We have then to consider

$$\int_C \frac{dt}{(t-z_0)^n(t-z)} \tag{1}$$

which may be written

$$\int_C \frac{dt}{(t-z_0)^n(t-z_0-\overline{z-z_0})} = \int_C \frac{1}{(t-z_0)^{n+1}}\left\{1+\frac{z-z_0}{t-z_0}+\left(\frac{z-z_0}{t-z_0}\right)^2+\cdots\right\}dt.$$

Restrict z to lie within a circle Γ of center z_0, within C. The series of functions of t which constitutes the integrand is then uniformly convergent in a region including the curve C and may be integrated term by term around this curve. Since $n \geqq 1$, it follows that the integral of each term is zero and hence (1) is zero within the circle Γ. But (1) represents an analytic function of z at all points within C. This function is therefore zero.

Again if $P_i(z)$ is the principal part arising from an essential singular point z_0 it is an infinite series of the form $\sum_{n=1}^{\infty} a_n/(z-z_0)^n$, which converges at all points of the plane except z_0 and converges uniformly in any region exterior to a circle of center z_0. Consequently the integral $\int_C P_i(t)dt/(t-z)$ may be found by termwise integration. By the foregoing proof the integral of each term vanishes. We conclude finally that $\int_C P(t)dt/(t-z)=0$ for all points within C. Hence $\phi(z)\equiv\bar{f}(z)$ within C and $\bar{f}(z)$ furnishes the analytic continuation of $\phi(z)$ as far as this is possible outside C.

Consider next a point z outside C. Since $\bar{f}(z)$ is analytic within C it follows that $\int_C \bar{f}(t)dt/(t-z)=0$. Hence $\psi(z)=(1/2\pi i)\int_C P(t)dt/(t-z)$. The reasoning already used shows that this integral may be found by termwise integration, a typical term being

$$\frac{a_n}{2\pi i}\int_C \frac{dt}{(t-z_0)^n(t-z)} = \frac{a_n}{2\pi i}\int_C \frac{dt}{(t-z_0)^n(t-z_0-\overline{z-z_0})}$$
$$= -\frac{a_n}{2\pi i(z-z_0)}\int_C \frac{1}{(t-z_0)^n}\left\{1+\frac{t-z_0}{z-z_0}+\left(\frac{t-z_0}{z-z_0}\right)^2+\cdots\right\}dt.$$

If, for the moment, we restrict z to lie outside a circle Γ' of center z_0 including C, then the series which forms the integrand converges uniformly and may be integrated term by term. The integral is zero for every term except one, viz.,

$$-\frac{a_n}{2\pi i(z-z_0)}\int_C \frac{dt}{(t-z_0)(z-z_0)^{n-1}} = -\frac{a_n}{2\pi i(z-z_0)^n}\cdot 2\pi i = -\frac{a_n}{(z-z_0)^n}.$$

This value holds at all points outside the circle Γ. But since the integral from which it arises denotes an analytic function at all points outside C the same expression gives its value at all these points. It follows that for all points outside C, $\psi(z) = -P(z)$. Moreover $-P(z)$ furnishes the analytic extension of $\psi(z)$ as far as this is possible within C.

We have finally that for all points in R at which f is defined, $f(z) = \bar{f}(z) + P(z) = \phi(z) - \psi(z)$, as the theorem states.

When the function $f(z)$ and the curve C are given, the method of the foregoing proof serves to determine the functions $\phi(z)$ and $\psi(z)$. Thus, if the curve C be taken as the unit circle and the region R as the complete z-plane, simultaneous values of f, ϕ, and ψ are given in the following examples.

(1) f analytic within C, $\quad \phi = f, \quad \psi = 0.$

(2) $f = z + \dfrac{1}{z}, \quad \phi = z, \quad \psi = -\dfrac{1}{z}.$

(3) $f = \dfrac{1}{z(z-2)}, \quad \phi = \dfrac{1}{2(z-2)}, \quad \psi = \dfrac{1}{2z}.$

(4) $f = \dfrac{z}{4z^2-1}, \quad \phi = 0, \quad \psi = \dfrac{-z}{4z^2-1}.$

(5) $f = \dfrac{z^2+1}{(4z^2+1)(z^2+4)}, \quad \phi = \dfrac{1}{5(z^2+4)}, \quad \psi = \dfrac{-1}{5(4z^2+1)}.$

(6) $f = e^z + e^{1/z}, \quad \phi = e^z + 1, \quad \psi = 1 - e^{1/z}.$

(7) $f = 2z^2e^{1/z}, \quad \phi = 2z^2 + 2z + 1, \quad \psi = 2z^2 + 2z + 1 - 2z^2e^{1/z}.$

(8) $f = \sin\dfrac{1}{z}, \quad \phi = 0, \quad \psi = -\sin\dfrac{1}{z}.$

(9) $f = \cos\dfrac{1}{z}, \quad \phi = 1, \quad \psi = 1 - \cos\dfrac{1}{z}.$

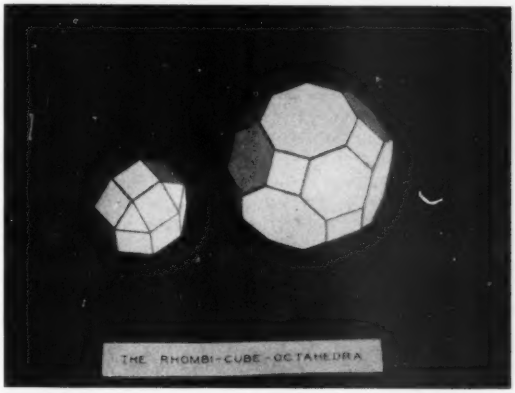
(10) $f = z\cos\dfrac{1}{z}, \quad \phi = z, \quad \psi = z - z\cos\dfrac{1}{z}.$

IV. Note on an Error in the Encyclopedia Britannica

By Michael Goldberg, Washington, D.C.

A list and description of the thirteen Archimedean solids (not including the prisms and prismoids), which are defined as polyhedra bounded by regular

polygons not all of the same species, and which have all the solid angles equal, is given on page 28, vol. 22 of the Encyclopedia Britannica (11th edition) under



the article "Polyhedron." The rhombi-cube-octahedra are incorrectly described. The small one is given as a polyhedron bounded by 12 *pentagons*, 8 triangles and 6 squares, while the actual figure is bounded by 18 *squares* and 8 triangles. The large one is described as a polyhedron bounded by 12 *decagons*, 6 *squares* and 8 *triangles*, while the actual figure possesses 12 *squares*, 6 *octagons* and 8 *hexagons*.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED*

The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems. By GRIFFITH CONRAD EVANS. New York, The American Mathematical Society, Colloquium Publications, Volume VI, 1927. viii+150 pages. Price $2.00.

"The work gives a unified treatment of the basis of the theory of Laplace's equation in two dimensions, suitable, it is hoped, for graduate students of a moderate degree of advancement, and is intended to be of service in the development of the theory of partial differential equations of elliptic type. These developments are generating a compound of two of the most important elements of modern analysis—the concepts of Lebesgue on the one hand, and of Volterra on the other."

Algebraic Arithmetic. By ERIC T. BELL. New York, The American Mathematical Society, Colloquium Publications, Volume VII, 1927. iv+180 pages, Price $2.50.

"There is a large, growing and somewhat uncoordinated body of results, with many aims and methods peculiar to itself, which falls into neither the classic theory of numbers nor the modern developments of the analytic and algebraic theories, and this region, which we have called algebraic arithmetic,

* The quotations given below, unless otherwise specified, are from prefaces and introductions to the books named.

offers many suggestive opportunities for systematic exploration. It is the purpose .... to outline a few promising directions in which progress may be made toward classifying, extending, and generalizing the methods and results of algebraic arithmetic."

Spherical Harmonics, An Elementary Treatise on Harmonic Functions with Applications. By T. M. MACROBERT, M.A., D.Sc., Professor of Mathematics in the University of Glasgow. New York, E. P. Dutton and Company, 1927. xii+302 pages. Price $4.50.

"..... A text-book on the elements of the theory of the spherical harmonics, with applications to mathematical physics, so far as this could be done without employing the method of contour integration, [and including] discussions on similar lines of Fourier series and Bessel functions, with corresponding applications."

The Pythagorean Proposition, its Proofs Analyzed and Classified, and Bibliography of Sources. By ELISHA S. LOOMIS. Given by the Author to the Masters and Wardens Associations of the 22nd Masonic District of the Most Worshipful Grand Lodge of Free and Accepted Masons of Ohio and Published by said Association. 214 pages. Price $2.00.

"The object of this work is to present to the future investigator, under one cover, simply and concisely, what is known relative to the Pythagorean Proposition, and to set forth certain established facts concerning the proof and the geometric figures pertaining thereto." Some 230 proofs of the theorem are given.

Análisis Matemático, por el Dr. HUGO BROGGI, Professor titular de las Universidades de La Plata y de Buenos Aires. Volumen II, Teorías Generales, Funciones de Más de Una Variable. La Plata, Facultad de Ciencias Físico-Matemáticas, 1927. Price 6 pesos.

This South American text deals with the following topics: planes and straight lines, determinants, quadric surfaces, complex numbers, simple differential equations, vector analysis, series, functions of more than one variable, curvilinear and multiple integrals.

Sir Isaac Newton. By C. D. BROAD, Fellow of the British Academy. Oxford University Press, American Branch, New York, 1927. 32 pages. $0.70.

This is the 1927 "Annual Lecture on a Master Mind" of the British Academy, and presents a survey of the life and the scientific work of Newton.

REVIEWS

History of the Sciences in Greco-Roman Antiquity. By ARNOLD REYMOND, translated by Ruth Gheury de Bray. New York, E. P. Dutton, n.d. [1927] viii+245 pages. Price, $2.50.

This is a translation of a work which appeared in 1924 under the longer title *Histoire des Sciences Exactes et Naturelles dans l'Antiquité des Écoles et des Principes*. Each edition has the honor of having a preface by M. Brunschvicg, "membre de l'Institut" and a scholar of standing. The author holds the chair of professor of philosophy in the University of Lausanne, and this work is based upon his lectures there and at the University of Neuchâtel. By reason of its sponsor, its author, and its translator it should rank as an authority, or at least be a safe guide for beginners. Reluctantly it must be said, however, that it is neither. What the French edition needed was not primarily a translator; it needed an editor. M. Reymond is a philosopher, but he is far from being a reliable historian; he has a general view of the development of the sciences, but his vision of details is imperfect.

As to the history of mathematics, his footnotes show only a slight acquaintance with secondary authorities, and in no case does he give evidence of having consulted any sources. He mentions Eisenlohr's edition of the Rhind Papyrus, but the text leads to the belief that he never read it; his translator has inserted a reference to Peet's edition, but it appeared after the French edition was written. He mentions the Moscow Papyrus, but expresses a doubt as to its existence at the present time, although his translator could easily have found that its contents are now known and that it is still accessible to scholars. He makes the assertion that the Egyptians were ignorant of the art of calculating an angle,—an ambiguous expression at best, and certainly untrue at its worst.

With respect to the accuracy of statement, both the French and the English versions show too many defects to allow them to pass unnoticed. To select one of a considerable number for example, Parmenides is given as having been born in 516 B.C. of which we are by no means sure; Zeno is given as born in 489, of which we are even more uncertain; and the latter is stated to have been twenty-five years younger then the former, which does not harmonize with the dates which M. Reymond gives as precise. If these were isolated errors, the matter would be unworthy of mention, but to express doubt as to the dates for writers like Ptolemy and to assign exact years for most of the other Greek scientists (generally quite uncertain), to give the dates 330–270 to Euclid, and to have Archimedes born in 257 B.C. instead of 287 (an error in the English edition only), is to go beyond the bounds of a historian's patience.

As to the translation, this is in the main very satisfactory. It is difficult, however, to see why "l'ancienne *philosophie naturelle* de l'Ionie" should appear in the much less acceptable form "the ancient *philosophy* of Ionia," or why "le rapport étroit" should have the uncompromising rendering of "inseparable connection." Such minor slips of the pen or of the typist as "Dox, 476" instead of "Dox., 476" are of course, not to be charged severely against any translator, however careful. Whether the French "On désigne souvent sous le nom d'hylozoïsme" should appear as "is often given the name Hylozoism," with the emphasizing capital and with no indication of the etymology of the "name" is a matter that may well be left to the judgment of the readers of this review. Since the Greek is not infrequently given in the text, it would seem to be appropriate to give the roots of such a word as this,—a word not so familiar in the history of science, no doubt, as in the history of philosophy.

It is quite unnecessary to dwell further upon particulars. Suffice it to say that the cases cited are, unfortunately, so typical as to warrant the judgment that, so far as it relates to mathematics, the work is unreliable and cannot be recommended for reference purposes or as a handbook to be used by students.

DAVID EUGENE SMITH

The Byzantine Astrolabe at Brescia. By O. M. DALTON. London and New York, Oxford University Press, American Branch. 14 pages+3 plates. Price 70 cents.

This memoir, communicated to The British Academy on July 7, 1926, describes a Byzantine astrolabe which was presented to the Museo dell' Età Cristiana at Brescia, Italy, by Signora Sajler, in 1844. The instrument bears a date equivalent to 1062 of our era. It is therefore contemporary with that celebrated Neoplatonist, Michael Constantine Psellus (1020–1110), whose works on mathematics were often printed in the sixteenth century and which, up until now, gave us the only knowledge that we had of this science in Constantinople in his time. The instrument, indeed, might have been conjecturally ascribed to him had it not been for the fact that one of the inscriptions relates that it was made by "Sergius the Persian," a man of consular rank, and that another states that it was made under the "decree and command of Sergius."

The author adds to the value of his study by giving a brief description of astrolabes in general and a reference to earlier Greek writers upon whose works Sergius or his craftsman may have depended,—chiefly Joannes Philoponus[1] and Severus Sabokt[2] (Sebokht, Sabocht), bishop of Kennesrin (Quenšerin, Kenneshre), on the Euphrates each of whom depended upon writers like Ptolemy, Proclus, and Ammonius, and upon the poet-orator Synesius, who had studied under Hypatia.

The inscriptions are all in Greek and although Sergius speaks of himself as a Persian the instrument gives evidence of only the Byzantine influence. The significance of the piece is thus summarized by the writer:

"Before the examination of the Brescia astrolabe, it was possible to question the activity of medieval Byzantine astronomers, but this is a position which can no longer be maintained. Sergius cannot have been the sole ἐπιστήμων[3] of his age and city, nor can the craftsman who worked for him have been the single Greek maker of mathematical instruments. . . . . The astrolabe of Sergius . . . . proves that Hellenistic science was remembered as late as A.D. 1062 and makes it probable that astronomical studies continued until the time of the Fourth Crusade. This instrument, inscribed and precisely dated, takes its place as a historical document; it has all the evidential value of a manuscript."

DAVID EUGENE SMITH

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[1] The author says that he "lived in the first half of the seventh century," which is probable, but the date commonly assigned (*c.* 640) is quite uncertain, as is also the century.

[2] He lived in the first half of the seventh century.

[3] In the inscription the word is ΕΠΙCΤΙΝΟC, which the writer conjectures to stand for ἐπιστῆμονος, a savant.

Modern Plane Geometry. By J. R. CLARK and A. S. OTIS. Yonkers, World Book Company, 1927. x+310 pages.

Plane Geometry. By D. M. BERNARD. Richmond, Johnson Publishing Company, 1927. xv+334 pages.

The first of these books, unlike English books with similar titles, does not deal with topics in plane geometry developed in recent times. The book is modern in method rather than in content, and contains the standard topics of the recognized secondary school course. But it is emphatically not just another plane geometry text-book. "In this book," the authors claim, "you will find for the first time, carefully planned and fully prepared, that development of each topic which psychology teaches and which your experience has shown you to be essential to the complete understanding of the statement and proof of a proposition." The authors further assert that tests in some twenty schools, in which experimental editions of their book were brought into competition with old style texts giving complete synthetic proofs, showed their book "significantly superior in developing *power*" and "slightly superior in developing *information*." In the hands of skillful, wise and sympathetic teachers, it seems probable that this record will be maintained in the larger field now open.

This greater efficiency has been secured by giving the student a different attitude than is common towards the work before him. The authors consistently place the student in the position of a discoverer, not only of proofs but of facts. "Do you think that if a triangle has two equal angles, the sides opposite those angles are equal?" "Ernest reasoned as follows . . . . . Do you see any error in the reasoning?" The student is also met throughout by a series of *challenges*. These usually ask the student to prove a fact whose truth is suspected without the use of the next section in the book. If he succeeds he may give himself a grade of *A* in a table provided for that purpose at the back of the book. If he fails, he may read the next section for suggestions; if, with the help of these hints, he succeeds in devising a proof, he marks himself *B*. This system of self-classification is extended also to work with originals. Other distinctive features are found in this text including an excellent "introductory course" leading up to the demonstrative geometry. But perhaps enough has been said to indicate that the book possesses great merit and originality and deserves a thorough study by teachers. Not all of them will approve of all its features or find it adapted to their own methods of teaching the subject; but all can profit by its consideration.

The merits of this progressive text are so great, that overstatement in the preface may be forgiven. For the book, though a pioneer in modernism, is not entirely without precursors. For example, among books known to the reviewer, the *Elementary Geometry* by the English authors Godfrey and Siddons,

published in 1903 by the Cambridge University Press, featured a thing,—"systematic training in the discovery of geometric facts"—that Clark and Otis say "has not appeared in any previous text book;" and E. R. Smith in his *Plane Geometry developed by the Syllabus Method*, published in 1909 by the American Book Company, taught students to discover proofs, if not facts, for themselves.

The *Plane Geometry* by Bernard will appeal to the more conservative teacher. Its chief departure from the traditional book is that in its developed proofs, after the statement of what is given and what is to be proved, a brief outline of the proof is inserted, labelled *method*, before the proof is set out in detail. In many cases, the method is given without the detailed proof. While this seems like a small thing, the reviewer believes that here we have a real improvement. No doubt numbers of teachers have been encouraging their pupils to analyse the given facts and to see the reason for the steps used before learning the proof. This brief outline will be of great help here. Curiously enough, in those proofs which are given at length in the Clark-Otis book, exactly the same thing is done, the outline here being labelled *plan* instead of *method*. Bernard's book is characterized by a clear statement of proofs, good diagrams and an excellent selection of originals, including groups of supplementary excercises taken from entrance examinations at a number of colleges and credited to their source. Wherever possible the author has simplified his proofs by using single letters for angles and for line-segments. The theorems are not assigned numbers, other than those of the sections. Both books are attractively manufactured, meet every requirement as to content, and deserve well of teachers of mathematics in secondary schools.

J. W. CLAWSON

Integral Bases. By W. E. H. BERWICK. Being number 22 of Cambridge Tracts in Mathematics and Mathematical Physics, edited by G. H. Hardy and E. Cunningham. Cambridge University Press, 1927. 5+95 pages.

The algebraic integers in a field of algebraic numbers of degree n form an aggregate of the type

$$x_0\omega_0 + x_1\omega_1 + \cdots + x_{n-1}\omega_{n-1},$$

where $x_0, x_1, \cdots, x_{n-1}$ are rational integers, provided that $\omega_0, \omega_1, \cdots, \omega_{n-1}$ is a suitably chosen set of algebraic integers in the field. The n integers which form such an *integral basis* of the field are not unique but every two integral bases are connected by linear homogeneous relations of a certain particular type. A quadratic field can always be defined by an equation of the form $\theta^2 - m = 0$, where m is an integer which is divisible by no square factor greater than unity; and then an integral basis is of the form

$$(1, \tfrac{1}{2} + \tfrac{1}{2}\theta) \text{ or } (1, \theta) \text{ according as } m \equiv 1 \bmod 4 \text{ or } m \equiv 2 \text{ or } 3 \bmod 4.$$

Methods of finding an integral basis of a cubic field have been given independently by G. B. Mathews and G. T. Woronoj. Kummer has obtained an integral basis of a cyclotomic field, while Hilbert's detailed investigations of quadratico-quadratic fields include a discussion of the algebraic integers belonging to them. Beyond these cases little appears to have been done heretofore in developing methods for constructing integral bases for algebraic fields.

In this tract the author develops a method of finding all the algebraic integers of the field $[\theta]$ defined by a rational integral irreducible equation

$$a(\theta) \equiv \theta^n + a_1\theta^{n-1} + a_2\theta^{n-2} + \cdots + a_n = 0.$$

His researches arose from his investigation of the special case $\theta^n - a = 0$, the problem to which he devotes his final chapter XV which requires about one-third of the whole monograph for its treatment. In this particular problem there are twenty-three cases which demand discussion. In treating this special case the author was led to the methods which he has applied to the general problem named.

The first five chapters (pp. 1–16) are devoted to preliminaries, including (among other things) a summary of the tract and an account of Dedekind's theory of algebraic numbers. The new methods are discussed in chapters VI to VIII (pp. 17–35) and an integral basis of the field defined by the equation $a(\theta) = 0$ is treated in chapter IX (pp. 35–39). The rule for putting down an integral basis follows a process of approximation, the first three stages of which are worked out in the tract. In his preface the author says, "Failing cases exist but the approximations given are sufficient to cover nearly any numerical equation not specially constructed to defy them." The author states that he has carried through a fourth stage in the approximation but that it was not thought necessary to print this since it appears to lead no nearer to finality than the three stages which are developed in the tract. Many of the methods apply equally to a relative field, a fact of which use has been made in the treatment of the special case of chapter XV. In the course of the tract a few numerical examples are worked out to illustrate the power of the methods when applied to a concrete case. An integral basis of $[a^{1/n}]$ is obtained for all values of a and n.

R. D. CARMICHAEL

A Table of Interpolation Multipliers. By A. S. LITTLE. Boston, Financial Publishing Company, 1927. 29 pages.

If $A, B, C, \cdots$, are tabulated values of a function for equal increments of its argument, then

$$A_1 = B - A, \quad A_2 = C - 2B + A, \quad A_3 = D - 3C + 3B - A, \cdots.$$

are known as the first, second, third, $\cdots$, differences and in terms of them the Gregory-Newton interpolation formula is written

$$A + kA_1 + \frac{k(k-1)}{2!}A_2 + \frac{k(k-1)(k-2)}{3!}A_3 + \cdots.$$

If it be assumed that all differences above the third vanish, this formula may be rearranged in the form

$$a(k)A + b(k)B + c(k)C + d(k)D.$$

When so written the multipliers containing k are independent of the function and so if tabulated would be usable for any interpolation problem involving only third differences. The author of this highly useful book has done precisely this work of tabulating these multipliers for $k = .002, .004, \cdots, 998$ and has arranged the whole in a very convenient form for use on a computing machine.

Unfortunately the book is published in a form which assumes its use only in the evaluation of bond purchase prices, and so is addressed to a non-mathematical public. A distinct gain would be made if a proper mathematical introduction were added and also the actual values of k were given in the tables.

C. F. CRAIG

Plane Trigonometry and Tables. By H. E. BUCHANAN and PAULINE SPERRY. Richmond, Johnson Publishing Co., 1926. xi+116 pages, tables 112 pages.

The subject matter in this book is presented in a clear, concise manner. There is no duplication of material and no essential facts are overlooked. The exercises are excellent.

Radian measure is introduced on the first page and is applied in problems throughout the book. The logarithmic solution of right triangles is taken up before the functions of the general angle. A clear explanation of logarithms and of the tables is given in the text, together with enough facts and illustrations to enable the average student to comprehend what is meant by accuracy of results. The tables are quite satisfactory though we wonder why the three place table of logarithms of numbers is carried only to 500 instead of to 1000. There is, of course, a five place table of logarithms of numbers 1 to 10,000.

In dealing with general angles, identities, etc., the book follows the usual procedure. The symbol ∞ is used to indicate "a state of affairs, and is not a symbol for a number." One always hopes in vain that a new text in trigonometry will state the simple truth, that tan 90° does not exist. The proof of the addition formulas by projection seems to offer some advantages over the usual method, especially in the case of obtuse angles.

The chapter on the solution of oblique triangles is short and to the point. The student is not burdened with a mass of unnecessary formulas. All cases

are covered. The applications to polar coordinates and to wave motion are well presented. We would prefer to see the curves in cartesian coordinates drawn for negative values of the abscissa as well as for positive values.

P. A. FRALEIGH

Plane Trigonometry with Tables. By M. A. KEASEY, G. A. KLINE and D. A. MCILHATTEN. Philadelphia, P. Blakiston's Son and Co., 1927. 63 pages +tables. Price $1.28.

This text has been written with emphasis on the use of Trigonometry in Engineering and "contains all the necessary material for college preparation."

The problems in the text are numerous and of sufficient variety from the point of view of usefulness to impress the beginner with the value of the subject. They constitute , in the opinion of the reviewer, the best part of the text. The definitions of the first few pages are a good introduction to the subject. The tables consist of a five place table of logarithms of numbers from 100 to 1000, a five place table of the logarithms of the trigonometric functions at one minute intervals, and a four place table of the natural functions. There are no tables of proportional parts. The book is provided with a good index.

There is, however, a great lack of mathematical exactness, expecially in the treatment of the new functions from the point of view of analytics. The usual discussion of directed lines is omitted, the definitions of coordinates are incomplete, and in the figures lines are read indiscriminately in either direction. There are also many careless forms of expression which detract from the accuracy of thought.

If there is no desire to teach anything of trigonometry beyond the numerical solution of triangles, the book may prove of advantage; but unless one can be sure that the mathematics learned therefrom will not be required in later work, especially in analytics, the presentation seems unmathematical.

A. W. SMITH

Die Vektoranalysis und ihre Anwendung in der theoretischen Physik. By W. V. IGNATOWSKY. Third Edition. Teil I: Die Vektoranalysis, X+110 pp. Teil II: Anwendung der Vektoranalysis in der theoretischen Physik, IV+123 pp. Leipzig-Berlin, Teubner, 1926.

This is the third edition of Ignatowsky's book, which first appeared in 1909-10. Reviews of the first edition can be found in the Bulletin of the American Mathematical Society (November, 1910, p. 100) and in the Jahrbuch über die Fortschritte der Mathematik (bd. 41, h. 1, 1910). No substantial changes are made in this new edition, except a few minor additions.

In chapter I, an article on reciprocal systems of vectors is added, which is

utilized in proving that the definitions of gradient, divergence and curl are independent of the form of the volume.

In chapter IV, a change is made in the proof that a vector function can be represented as a sum of a curl and a gradient. In the chapter on dyadics, the sections on the relations between polar and axial vectors are replaced by detailed discussions of linear vector functions, tensors, etc.

The theory of linear vector functions is then applied, in vol. II, to the discussions of the properties of elastic bodies. On the whole, the book ought to make profitable reading for students in applied mathematics.

S. D. ZELDIN.

Die Ebene Vektorrechnung und ihre Anwendungen in der Wechselstromtechnik. By H. KAFKA. Teil I: Grundlagen, VII+132pp. Leipzig-Berlin. Teubner, 1926.

This book is mainly intended for students in electro-technics. It treats the theory of complex quantities from a geometrical standpoint and their applications to electrical problems. The book ought to appeal to the electrical engineer, since it gives him something concrete to work with. In the first chapter plane vectors are defined, the operations of addition, subtraction and multiplication (scalar and vector) of vectors are discussed and amply illustrated by diagrams. The second chapter studies the sinusoidal time functions and their vectorial representation. The remaining chapters are devoted to the vector representation of principles in electro-magnetic theory.

S. D. ZELDIN.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activites should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB TOPICS

THE EVOLUTION OF NUMBERS—AN HISTORICAL DRAMA IN TWO ACTS*

By H. E. SLAUGHT, University of Chicago

ACT I: INTERNAL DISSENSION IN THE SECRET SOCIETY OF NUMBERS.
Time: 1625 A.D. Place: Vestibule of the Secret Chamber of the Society in Paris.
Dramatis Personae:
The Cardinal: (Representing Whole Numbers, the original Aristocrats).
The Barbarian: (Representing Negative Numbers, the First Interlopers).
The Arab: (Representing Zero, the Second Interloper).
The Hoi Polloi: (Representing Common Fractions).
The General: (René Descartes in Charge of Personnel).

Prologue to Act I. (*A young speaker dressed in heraldic costume stands between the parted curtains.*)
My fellow devotees of Mathematics, you will agree that the Number concept is as old as the human race. Let me recall to you some of the struggles through which this concept has come in order to

* This playlet may be found helpful in arranging entertainment features for undergraduate mathematics clubs.

reach its present development. In primitive times there were only a few whole numbers, used for counting, and a few special fractions each with numerator unity. The idea of negative numbers was the outgrowth of the experience of many centuries. It was Descartes in the seventeenth century who conceived the notion of picturing numbers by points on a line extending in both directions from a zero point and thus giving negative numbers and zero a concrete and natural representation. The aristocratic whole numbers then became Descartes' positive integers. They had maintained a secret organization, so to speak, for some 3000 years and had persistently black-balled the negative numbers every time they had sought admission. Likewise zero had been an outcast, a nobody, since the childhood of the race. Even the resourceful Greeks and the mighty Romans were deprived of renown which might have been theirs if they had only been wise enough to admit the humble zero to their clumsy number orgainzations. What they failed to do the wiser Hindus and Arabs accomplished later on when they brought back to Europe the Hindu-Arabic number symbols and the decimal notation—possibly the greatest invention of all time. Well might Descartes be awarded martial honors for his services as "general in charge of personnel" when he stationed the positive and negative integers and zero as sentinels on his Number Line and then filled in the intervening spaces with the lowly fractions, now recognized as swarming in countless hordes, thus forming the battle front of the mathematical warfare which was to conquer the world.

(*The speaker withdraws from the stage and the curtains open. The Barbarian enters and tries to open the door to the adjoining room, the Secret Chamber of the Society of Numbers. The Cardinal then enters and finds the Barbarian trying to break in. Each actor wears a placard on his breast showing his name in large print.*)

The Cardinal (*Dressed in a cardinal robe*): Is this you, the Barbarian, trying again to gain admittance to the secret chamber of our Number Society?

The Barbarian (*Dressed in a rakish costume*): Yes, General Descartes has told me that we are fully entitled to a place coordinate with you on his new Number Line.

The Cardinal: This is an outrage. We, the Cardinals, have held the major positions in our Society for 3,000 years and it is preposterous now for General Descartes to assign us to positions on the right wing and you to equally important positions on the left.

The Barbarian: But you Cardinals cannot hold the modern battle line alone. It may have been simple enough a few centuries ago when your Number Society was small and there was no need for an extended battle front.

General Descartes (*Entering in a costume appropriate to the time. See the Open Court Portraits of Famous Mathematicians*): The Barbarian is quite right. The number army has been only half organized heretofore and many a battle in mathematics has been lost solely because the army had no left wing. What would you think of an army in the military world that should try to do its fighting with one wing while the enemy was free to execute a flank movement on the unprotected side?

The Barbarian: As an example, there was Captain Cardan a hundred years ago, who needed us Negative Numbers the very worst way in order to complete his victory over the Cubic Equation. We were there ready and anxious to do our part but he spurned our assistance and called us "fictitious" and "absurd."

The General: You Cardinals and Barbarians must quit your quarreling and divide the responsibility of sentinel duty on the Number Line, the Cardinals on the right wing and the Barbarians on the left.

The Arab (*Now entering clad in typical Arab costume*): Yes, but that leaves an unguarded sentinel post between the wings and I alone, the long despised and rejected Zero, am the only one qualified to guard this point. I am absolutely the center of things, the *King Pin*, so to speak, in this whole army formation.

The Hoi Polloi (*Clad in midget costume, entering just as the Arab ceases speaking*): But an army must have something more than King Pins and Sentinels! After all, the hordes of Common Fractions are the ones who constitute the real battle front and we are the only ones qualified for this service.

The Cardinal: Well, if we *must* divide the honors with these Barbarians and the Arab, then our humiliation is complete. We are aristocrats no longer.

The General: But World Progress demands it. You have blocked the world's business for 3,000 years by persistently black-balling these applicants for admission to your Society. Take the Arab, for instance. Without him the ancient Greeks with all their boasted culture could do but little in arithmetic. The mighty Romans could conquer the world in war but they could not master Arithmetic without the help of Zero. It remained for the Hindus and the Arabs to show the world a decimal number system in which Zero was indeed the "king pin." The Arabs brought this wonderful system to Spain and Leonardo of Pisa introduced it to the rest of Europe; but it took Europe three centuries to fully appreciate this great contribution of the East to the West. Now the Arab and the Barbarian shall come into their own. They shall occupy positions of honor on the Number Line and you Cardinals shall recognize them as your equals. You think it will detract from your glory, but, quite the contrary, it will be your making. You have been sitting around for 3,000 years doing practically nothing but teaching people how to count, while the world's great problems have gone unsolved.

The Cardinal: We recognize our great short-sightedness in this matter and we will now follow your directions.

The General: And another thing! This secrecy business must stop. It began way back in Egypt when Ahmes, the Priest, wrote in the minutes of his famous Papyrus "directions for knowing all dark things." This sort of thing was continued down through the Centuries. Some one would solve a problem and carefully conceal the method and then challenge some opponent to find the solution. Even Cardan is said to have gotten his solution of the Cubic from Tartaglia under false pretenses with a promise never to give it away—which promise he deliberately broke.

The Cardinal: Yes, the vow of secrecy was a mistake. We will hereafter do our work in the open.

The General: Finally, you must adopt a Constitution and By-Laws and then examine each applicant for membership on his merits and not act on mere sentiment and prejudice, as you have done all these centuries with the Barbarian and the Arab. The only criterion for membership must be obedience to the By-Laws and allegiance to the Constitution.

The Cardinal: We will do these things, but what about the Arab, the Barbarians, and the Hoi Polloi? Will they also conform?

All in chorus: Yes, we will conform—so say we all of us.

(*Two attendants now enter bearing an exhibition of a well organized Number Line, consisting of a strip of wrapping paper ten or twelve feet long, on which a heavy line is marked off with units and labelled with large figures, positive, negative and zero. The Cardinal, the Barbarian, Zero, and the Hoi Polloi stand back of this Number Line, while General Descartes stands in front with a pointer indicating the respective subdivisions.*)

The General: Henceforth you Cardinals will continue to stand as sentinels on the right wing, and you Barbarians will do duty likewise on the left wing; while the Arab will stand guard at the key position between the two wings, and the Hoi Polloi will fill in the open spaces between the sentinel posts. Thus is the Number Line ready to attack the world's great problems.

(*The Curtain Falls*)

ACT II: INTERNATIONAL ARMISTICE AND PEACE IN THE NUMBER WORLD.

Time: 1825 A.D. Place: International Open Forum.

Dramatis Personae:

The Irrational, $\sqrt{2}$,: (The Third Interloper)
The Complex, $\sqrt{-1}$,: (The Fourth and Last Interloper).
The Continuum: (The Solid Front Number Line).
Professor Dedekind: (Representing the Great Divide).
Professor Gauss: (Representing the Complex Plane).

Prologue to Act II. (*Same speaker as in the Prologue to Act I, but now grown old, standing between the parted curtains.*)

My fellow devotees of mathematics, it is now 200 years since General Descartes organized his famous Number Line in which the aristocratic whole numbers were compelled to share the honors of sentinel duty with the negative numbers and zero. During these two centuries a lot of things have happened. When the vow of secrecy was removed from the Number Society as Descartes commanded, there was an immediate renewal of the attempt of the Irrational and the Complex to break into the system. Their struggle for recognition had already been going on for at least 600 years, but encouraged by the success of the Barbarians and the Arab in the seventeenth Century, they then battered harder than ever at the doors of the Number Society till the beginning of the nineteenth century when strong friends like Professors Dedekind and Gauss came to their aid and finally secured their admission. You are now to hear these actors in the Number Drama speaking for themselves.

(*The herald withdraws and the curtain is opened showing Professors Dedekind and Gauss, in costumes befitting the time* (*See the Open Court Collection of Portraits of Mathematicians*), *walking and talking together.*)

Professor Dedekind: You recall a bold program proposed by General Descartes two hundred years ago when he commanded the Society of Numbers to abolish secrecy and to adopt a Constitution and By-Laws.

Professor Gauss: Yes, I recall the story. It was, indeed, a bold program and it has taken just about two Centuries for its accomplishment. Oh, well, that is fast work compared to the slow reforms which General Descartes brought to a climax when he compelled the Cardinals to admit the Barbarians and the Arab to full fellowship in the Society.

Dedekind: It is less than ten years ago since those by-laws were fully agreed upon and published, namely, the Commutative, Associative and Distributive Laws which really govern our Number System.

Gauss: It took a long time for people to realize that these Laws do actually control the operations of our Algebra and that they are the only criteria for admission to the Society of Numbers.

Dedekind: Of course the Cardinals were the first to adopt these By-Laws, but it took a long time for people to see that the Barbarians, the Hoi Polloi, and Zero all conform to these same Laws, and hence actually belong to the Society just as much as the Cardinals.

Gauss: It does seem awfully slow progress as we look back upon it now, but when we realize how far we have come since Ahmes, the Egyptian Priest, started the organization 3500 years ago, it seems that we have done wonders in these last two hundred years.

Dedekind: Well, it is up to us to see to it that things move faster in this 19th Century than they did in the preceding centuries. It was a shame and a disgrace to Europe that the Hindu-Arabic symbols and the decimal notation which were brought to us from Arabia by way of Spain 600 years ago should have gone almost unnoticed for three centuries before we began to awaken from our long sleep of the Dark Ages to see that Asia had put into our hands what was destined to become the most powerful instrument ever devised by the mind of man.

Gauss: Yes, a great responsibility rests upon us of this generation. Already the members of the Society are getting busy. The Constitution is clear as to our duty. It states that all hands shall add, subtract, multiply, divide, raise to powers and extract roots unceasingly, in season and out of season. That is our job. Thus shall the world's business be transacted.

Dedekind: It is interesting to note that the first and only ammendment to the Constitution had to be enacted almost at the very start, namely, that division by zero shall be forever prohibited.

Gauss: Yes, I recall how some well-meaning people have tried to divide by zero and have thought that the quotient was a member of our Society, but have finally been convinced that none of our members can act in that capacity.

Dedekind: There is another curious phenomenon which has caused a lot of trouble in the last two centuries, namely, when our members began to extract roots promiscuously they soon ran across

such an anomalous case as the square root of 2 and found that it was neither a Cardinal nor one of the Hoi Polloi. Hence they naturally concluded that such a thing could not belong to our Number Society and so they refused its admission. This case has caused me a lot of trouble and anxiety, but I believe that I now have it satisfactorily explained.

Gauss: That is no more curious than the case that I have been worrying about, namely, the square root of negative one. These two interlopers have caused the Society more trouble than all the orthodox members put together. They have repeatedly applied for admission, and have been consistently rejected for many centuries. Here they come now. Let us hear what they have to say for themselves. (*The Irrational and the Complex now enter, in costumes appropriate to the age, walking and conversing*).

The Irrational: I surely have had a rocky time of it. I have tried repeatedly to break into the Number Society but have always failed. They have even told me that I am nobody and have no respectable standing at all. But even Pythagoras knew better than that 2300 years ago. He saw that I represent the diagonal of the unit square and that I am just as respectable as the side of the square. Bhaskara in India 600 years ago recognized me as a thoroughly respectable citizen, and the Englishman, Robert Recorde, 300 years ago even wrote a book in which he paid sincere respect to me. But it remained for Professor Dedekind and others of the present time to really vindicate my rights and to atone for the injustice which has been done to me for three thousand years. I want to see him and thank him in person.

The Complex: If you think you have had a rocky time, what do you think of me? I did not have *even one* friend and defendant to do for me what Pythagoras did for you. Not one of the Indian members stood up for me as Bhaskara did for you. I was absolutely an outcast till just 200 years ago when Albert Girard recognized me as a legitimate citizen, while he was solving equations. It is true that Captain Cardan, 100 years earlier, had met me when solving his cubic but he said I was "curious" and "useless." So things went on until a Frenchman by the name of Argand wrote an essay a few years ago in which he defended me and showed why I should be admitted to the Number Society. But nobody paid any serious attention to this essay or to me till Professor Gauss put me on the map and I am hoping to see him and thank him personally for thus atoning for the three thousand years of injustice which *I* have suffered.

The Irrational (approaching Professor Dedekind): Here is Professor Dedekind now. Allow me to thank you kind sir, for what you have done for me. You cannot know what a relief it is to be admitted to the Number Society after thirty centuries of exile. I know that other friends have been working in my behalf, such as Professors Cantor and Weierstrass, but I am sure your name will go down in history as my emancipator and that the "Dedekind Cut" by which my shackles were broken will forever stand as one of the greatest achievements of mankind. Incidentally you will be pleased to know how gracefully and deferentially the Hoi Polloi have separated themselves into two groups at the "Cut" in order to allow me to take my place on the Number Line. Again, kind sir, let me thank you from the bottom of my heart.

Professor Dedekind: You overwhelm me with your words of praise and gratitude. I assure you that it has been one of the greatest joys of my life to be able to render you this service, especially when I realize that you represent an untold multitude of Irrationals, including the famous π and e, all of whom are now given places on Descartes' Number Line. You have my blessing. Go and take your part in the World's Work.

The Complex (approaching Professor Gauss): Ah! Here is Professor Gauss. I am so glad of this opportunity to meet you and to thank you in person for the wonderful way in which you have opened up life's opportunities for me. It was bad enough to be an outcast for three thousand years, but it was the last straw when my friend and companion, the Irrational, was finally given a "place in the sun" on Descartes' Number Line while I was told that there was absolutely no room left there for me.

But just as I was about to die of despair, your brilliant invention, the Gauss Complex Plane, rescued me from oblivion, and now I not only have my place in the sun but I am actually Lord and Master of the whole Number Field, of which the Descartes' Number Line is only a small and very special part. I want you to know, kind sir, that I appreciate this great honor and that I am fully

conscious of the tremendous responsibility which this wide-open Number Field places upon me. Again, kind sir, let me thank you from the bottom of my heart.

Professor Gauss: You have, indeed, been most patient and long-suffering but your long-delayed recognition has been due not so much to the prejudice of the older members of the Society as to their ignorance. They did not know, and I suppose they could not know, the great possibilities for service that your admission to the Number System will open up. I firmly believe that the 19th Century will justify our highest hopes and will, indeed, see you enthroned as Lord and Master of the whole Number Field.

The Continuum (A sturdy fellow bearing a placard on which a section of the Number Line is very closely divided to represent the myriads of real numbers): At last the struggle of mankind to build a Number System adequate to do the world's business is crowned with success. There have been many scenes of conflict in building this System. There have been jealousies, secret plottings, black-ballings, banishments, but now the mathematical world is at peace. There is complete international understanding and only healthful rivalry. As for myself, I am the symbol of complete union and harmony on the real Number Line. I represent the Solid Front that the Society of Numbers presents to the world. We are now fully prepared to underwrite the world's great problems.

(*The drama closes with all the members of the Company on the stage beneath a banner on which is a graphic representation of the Real and Complex numbers. General Descartes and Professors Dedekind and Gauss are standing in front of the number Group, and all join in singing a stanza of "Hail, hail, the gang's all here."*)

CLUB ACTIVITIES

The Mathematics Club of the University of Oklahoma, Norman, Oklahoma.

During the year 1926–1927, the officers of the club were: First semester—Rothwell Stephens, president; Helen Meister, vice president; L. D. Montgomery, secretary-treasurer. Second semester—Max Heaslet, president; Narcissa Bond, vice president; Ruth Jenkins, secretary-treasurer. The meetings were held fortnightly on Thursday afternoons and refreshments were served before the beginning of the program.

The following topics were discussed:

October 26, 1926. "Welcome" by Dr. Reaves, Dean. "Practical value of mathematics" by Dr. Hassler. "Mathematical research" by Dr. Court. "Possible work of mathematics club" by Dr. Meacham.

November 18. "History of analytic geometry" by Miss Rivers. "Axioms of metrical and projective geometry" by Helen Meister.

December 9. "Excerpts of winnowed wisdom" by Wilma Gorton. "Ancient systems of numerical writing" by Eddie Eaves. "A mathematical story" by Narcissa Bond.

January 13, 1927. "Certain numerical curiosities" by Norma Pate. "Prime numbers" by Professor Townes.

February 10. "The life of Euclid" by Eunice Lewis. "How to find the day of the week having given the month, day of month, and year" by Charles Edmondson. "The development of pi" by L. D. Montgomery. "Multiplication on the fingers" by Dr. N. A. Court.

March 3. "Geometric constructions and paper folding" by Cleo Shaul. "Axioms applied to equations" by Mr. Paine.

March 24. "Non-Euclidean geometry" by Dr Elsie McFarland. "Permutation and combination" by Houston Roane.

(Report by Mr. L. D. Montgomery)

Mathematics Club of Rockford College, Rockford, Illinois.

Organized in October, 1921.

Purpose: "To further interest in the field of mathematics and to provide an opportunity for the presentation of other interesting phases of mathematics and allied subjects not taken up in the class room."

Meetings are held once a month. The officers for 1926–1927 were: Mary Larmore ('27), president; Mary Burchfield ('27), vice-president; Hazel Kelly ('28), secretary-treasurer.

Program for the year 1926–1927:

October 1, 1926. Special meeting at which time it was decided to take in new members at the first regular session.

October 27. Sixteen new members were initiated. The purpose of the club and the plans for the year were discussed. Mathematical puzzles furnished entertainment. Refreshments were served.

November 18. Professor McGavock gave an interesting talk entitled: "If we were on the moon, what we would see." The program illustrated the relation of mathematics and astronomy.

December 15. A program illustrating the relation between mathematics and religion. Talks on the subject were given by Margaret Lauritzen ('27) and Ruby Tjaden ('27).

January 9, 1927. A paper written by True Kimbal ('30) on "Mathematics and architecture" was read.

February 16. The program showed the relation of mathematics and music. Katherine Bogardus ('27) gave a talk on this subject. Ruth Goller ('27) played piano selections and Dorothy May Anderson ('30) gave a cornet solo.

March 16. A talk on mathematics and physics was given by Professor Herrick. She said that mathematics was the language of physics.

April 18. Open meeting held in the Chapel. Dr. Miller gave a very enjoyable lecture on "Mathematics; a bridge between two generations." Refreshments were served.

May 2. Annual mathematics club banquet. The program consisted of a geometry book supposedly written by Dr. Miller. It contained many axioms and theorems concerning the future of the members of the club.

May 19. The relation of mathematics and English. Carolyn Keyt ('27) reviewed an article on the subject and Frances Hills ('29) read an interesting paper she had written. Mildred Greenlee ('27) gave statistics on the number of times mathematical terms were used in literature. The program was concluded by solving mathematical puzzles.

Officers for 1927–1928, elected at this meeting, are: Hazel Kelly ('28), president; True Kimbal ('30), vice-president; Dorothy May Anderson ('30), secretary-treasurer.

(Report by Miss Dorothy May Anderson)

The Mathematics Club of the University of North Carolina, Chapel Hill, N. C.

The Mathematics Club of the University of North Carolina for the session 1926–1927 opened its first meeting October 20, 1926, with membership limited to undergraduates who had continued in mathematics beyond their required work. The following officers were elected for the ensuing year: President, E. A. Cameron; vice-president, J. L. Ring; secretary-treasurer, Dan Hall; faculty adviser, Dr. J. W. Lasley, Jr.

The Club met at 7:30 P.M. as follows:

October 20, 1926. Election of officers and a talk by Mr. M. A. Hill.

November 17. "The maximum triangle in an ellipse." First method, E. A. Cameron; second method, J. L. Ring.

December 15. A social and a lecture on mathematical fallacies and puzzles by Dr. J. W. Lasley, Jr.

January 19, 1927. "Extended synthetic division" by W. S. Barney. "The apothem of a regular pentagon from a geometrical standpoint" by C. W. Hook.

February 16. "Duplication of the cube." First method, E. A. Cameron; second method, C. D. Whisnant. "History of the duplication of the cube" by Alton P. Hall.

March 16. A social and a talk by Mr. L. L. Garner.

April 18. "The trisection of an angle" by Dan Hall. "History of the trisection of an angle" by J. E. Johnson.

May 23. A social and a talk by Prof. E. T. Browne.

(Report by Mr. A. P. Hall)

The Euclidean Circle of the University of Indiana, Bloomington, Indiana.

The officers for year 1926–1927 were: President, Lehman Shugart; vice-president, treasurer, Mary Francis Atlerton; secretary, Arbys Elizabeth Roberts; faculty advisors, Professors Rothrock, Davis, and Hennel.

Program for the year:

November 15. "Magic cubes and squares" by Richard C. Schneider.

December 6. "Three martyrs of science" by Harold T. Bentley.

December 20. Social Meeting—Memorial Hall.

January 10. "Recent astronomical discoveries" by Prof. K. P. Williams.

March 21. In commemoration of Sir Isaac Newton. 1. Addresses by Professor S. C. Davisson and K. P. Williams of the department of mathematics. 2. Address by Professor W. A. Cogshall of the department of astronomy. 3. Address by Professor Mason E. Hufford of the department of physics.

March 7. Social and Initiation.

April 18. "Conjugate functions" by William J. Kirkham.

May 2. "Generatrix functions" by Irene Price.

May. Picnic.

(Report by Professor H. T. Davis)

The Mathematics Club of Columbia University, New York City.

The following meetings were held by the undergraduate mathematical club of Columbia University during the year 1926–1927:

1. "Geometrical fallacies" by Leonard Price.
2. "Number pegs" by A. R. Mosler.
3. "Mascheronian constructions" by P. Roseman.

(Report by Professor W. B. Fite)

The Mathematical Club of Harvard University, Cambridge, Mass.

During the year 1926–27, the officers of the club were: Mr. Robin Robinson, president; Mr. Morris Marden, secretary-treasurer.

The programs of the Club for 1926–27 were as follows:

October 20, 1926. "Riemann's theory of algebraic functions and functions related to them" by Professor W. F. Osgood.

November 3. "The skew square" by Mr. S. B. Sommerville.

November 17. "Topics in the calculus of variations" by Mr. T. L. Smith.

December 1. "European mathematics of to-day" by Professor G. D. Birkhoff.

December 15. "Legendre's normal forms and some applications" by Mr. T. R. Long.

January 5, 1927. "Motor calculus" by Mr. C. I. Lubin.

January 19. "Multiple integrals with singular integrands dependent on a parameter" by Mr. A. B. Brown.

February 16. A joint meeting was held with the Harvard Engineering Society. Dr. Joseph Slepian of the Westinghouse Electric Company spoke on "Some mathematical ideas in electrical engineering."

March 2. "An application of theta functions in the theory of numbers" by Mr. C. N. Liu.

March 16. "A recent treatment of maxima and minima" by Mr. S. S. Cairns.

March 30. "The determination of analytic functions" by Professor Philip Franklin of the Massachusetts Institute of Technology.

April 13. "The parallelism of Levi-Civita" by Mr. H. B. Hammat.

April 27. "Some circles related to the triangle" by Mr. G. B. Price.

May 11. "Entire functions" by Mr. F. C. Jonah.

May 25. "Parallelism in Finsler space" by Dr. Harry Levy. The winners of the Robert Fletcher Roger's prizes were announced as follows: Mr. C. N. Liu, first prize; Mr. G. B. Price, second prize.

The following officers were elected for the year 1927–28: Mr. Clarence I. Lubin, president; Mr. G. Baley Price, secretary-treasurer.

(Report by Mr. G. B. Price)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3310. Proposed by B. F. Finkel, Drury College.

Find the envelope of a system of circles having for diameters the secants of constant length, $2r$, of a conic.

3311. Proposed by A. Pelletier, Montreal, Canada.

Eliminate $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$ from the following equations:

$$a^{-2}x_1^2 + b^{-2}y_1^2 + c^{-2}z_1^2 = 1, \quad a^{-2}x_2^2 + b^{-2}y_2^2 + c^{-2}z_2^2 = 1, \quad a^{-2}x_3^2 + b^{-2}y_3^2 + c^{-2}z_3^2 = 1,$$

$$a^{-2}x_1x_2 + b^{-2}y_1y_2 + c^{-2}z_1z_2 = 0, \quad a^{-2}x_2x_3 + b^{-2}y_2y_3 + c^{-2}z_2z_3 = 0, \quad a^{-2}x_1x_3 + b^{-2}y_1y_3 + c^{-2}z_1z_3 = 0,$$

$$a^{-2}x_1x + b^{-2}y_1y + c^{-2}z_1z = 1, \quad a^{-2}x_2x + b^{-2}y_2y + c^{-2}z_2z = 1, \quad a^{-2}x_3x + b^{-2}y_3y + c^{-2}z_3z = 1.$$

3312. Proposed by J. V. Uspensky.

For every prime $p=8l+3$, a root of the congruence $\xi^2 \equiv -2 \pmod{p}$ is given by

$$\xi = l!\,(5l+1)!/2l! \pmod{p}.$$

For every prime $p=12l+7$, a root of the congruence $\xi^3 \equiv -3 \pmod{p}$ is given by

$$\xi = C_{6l+3}^{l}/C_{6l+3}^{3l+1} \pmod{p}.$$

3313. Proposed by L. S. Johnston, Pennsylvania State College.

Consider the infinite sequence $[a_n]$ of real positive numbers with the recurrent relation $a_{k+1}^2 = 2a_k/(1+a_k)$ $(k=1, 2, 3, \cdots)$.

(1) Prove $L_{k\to\infty} a_k = 1$ for every a_1.

(2) Prove $L_{n\to\infty} \Pi_{k=1}^{n} a_k$ exists and is different from zero for every a_1.

(3) Express the limit in (2) as a function of a_1.

The problem is of course trivial for $a_1=1$.

3314. Proposed by J. B. Reynolds, Lehigh University.

A uniform flexible chain of length a, weight w, slides down an arch of a smooth inverted cycloid of total length $8a$ in a vertical plane with its vertex lowest. If the chain is released from rest in a position on the arch in which one end is at one end of the arch, find the time until the middle of the chain is at the lowest point of the cycloid, the speed of the chain at that instant, and the tension at its middle point then.

UNSOLVED PROBLEMS

Readers are requested to send in solutions for the following unsolved problems which were proposed in the years 1921, 1922, 1923. The number of each problem is printed in bold face type, with the page number following.

1921

2875, 37; **2882,** 90; **2883,** 139; **2886,** 139; **2888,** 139; **2889,** 139; **2891,** 184; **2905,** 277; **2909,** 326; **2911,** 326; **2915,** 327; **2918,** 327; **2922,** 392; **2926,** 393; **2928,** 466; **2935,** 467; **2937,** 467; **2939,** 468; **2940,** 468.

1922

2946, 29; **2952,** 81; **2953,** 81; **2954,** 81; **2956,** 81; **2960,** 129; **2961,** 129; **2962,** 129; **2963,** 129; **2964.** 129; **2968,** 179; **2969,** 179; **2970,** 179; **2975,** 225; **2977,** 270; 270; **2984,** 314; **2985,** 314.

1923

3004, 76; **3006,** 76; **3007,** 76; **3009,** 76; **3012,** 146; **3015,** 146; **3016,** 146; **3018,** 205; **3020,** 206; **3026,** 275; **3028,** 275; **3031,** 275; **3033,** 276; **3034,** 276; **3038,** 337; **3045,** 449; **3048,** 450.

SOLUTIONS

3250[1927, 156]. Proposed by Norman Anning, University of Michigan.

Give, in a form suitable for high school students, a proof of the theorem that when the perimeter of a triangle is given the area is greatest when the triangle is equilateral.

I. Solution by A. Pelletier, Montreal, Canada

The area A of a triangle may be expressed by the formula

$$A^2 = s(s-a)(s-b)(s-c).$$

Now, if the perimeter $2s$ is constant, the maximum of A^2, and also of A, will take place when the product $(s-a)(s-b)(s-c)$ is the greatest. Since the sum of the three factors is $3s-(a+b+c)=s$, a constant, their product is maximum when they are equal. This principle can be proved in a very simple way. Hence, $s-a=s-b=s-c$, or $a=b=c$.

II. Solution and Remarks by Otto Dunkel, Washington University

The proof below will be based on the following proposition which is proved in many texts on plane geometry.

Proposition. *Of all triangles having a given perimeter and a given base, the isosceles triangle with the other two sides equal has the greatest area.*

Let S be the area of any triangle with the sides a, b, c which are not all equal, but such that $a+b+c=3e$, the given perimeter. Let E be the area of the equilateral triangle of side e; then it will be proved that $S<E$. Let c be the longest side, so that $c>e$, and consider the isosceles triangle with base c and with each of the other two sides equal to $a'=(a+b)/2$. Then its area $S'\geqq S$ by the above proposition, where the upper sign is used if $a\neq b$. We now compare S' and E by constructing a half of each triangle. Draw the right triangle AMC so that $\angle M=90°$, $AC=e$, $AM=e/2$. Then $\angle A=60°$. Produce MA to A_1 so that $MA_1=c/2$. Lay off on CA the length $CK=AA_1$, then $AK=AC-A_1A=a'$. Draw the parallelogram $AKDA_1$, and take G on CA so that CG is bisected at K. Draw GP perpendicular to MC cutting it in P. From the form of triangle CGP it follows that $GP=GK=KC=DK=A_1A$. Also $KP=KG$. Hence DP is perpendicular to AC and to A_1D, since DKG and PKG are equal equilateral triangles. With A_1 as center and A_1D as radius describe a circle cutting the segment MC in C'. Since DP is tangent to this circle at D, $CP<CC'$. Draw A_1C' cutting AC in F, and $C'G'$ parallel to MA cutting AC in G'. Then $C'G'>PG=A_1A$, and A_1MC' is one half of the triangle of area S'. Since $G'C'>A_1A$, the area of $FG'C'$ is greater than that of the similar triangle FA_1A. Hence

$$E-S'=2[\text{area } CG'C'+\text{area } FG'C'-\text{area } FA_1A]>2 \text{ area } CG'C'.$$

We have then $E>S'\geqq S$, and the theorem is proved.

REMARKS. The treatment of this matter as given in our plane geometry texts cannot be regarded as a proof of the proposition as stated in this problem. In such texts the proposition above or a similar one regarding isosceles triangles is first proved, and then there follows a corollary sometimes worded about as follows: Of all triangles having the same length of perimeter the one of maximum area is equilateral. In this form of wording the facts stated have really been proved, but this falls far short of proving that the equilateral triangle with the given perimeter has the maximum area. In order to prove this last fact with the aid of the corollary it must be shown that there exist one or several triangles of some sort of shapes with the given perimeter such that they have the maximum area. This can be done by continuity considerations, but in this case a satisfactory treatment is likely to be unsuitable for an elementary text. It seems to be better and simpler to prove directly that the area of the equilateral triangle is greater than that of any other triangle with the same perimeter. The following process has been used.

Starting with any triangle which is not equilateral, we may consider an unending sequence of isosceles triangles formed by replacing two unequal sides of one triangle by two equal sides having the same total length as before and thus obtain the next triangle of the sequence. In such a sequence the areas increase and it may be easily shown that the triangles approach an equilateral triangle with the given perimeter, also that the areas approach a limit. But here we meet with a difficulty, it must be shown that the limit approached by the areas is the area of the equilateral triangle approached. Here again we may show this by continuity considerations but in this case also such considerations would be unsuitable for elementary treatment. Such a process was used by Lhuilier, as stated by Steiner in some preliminary remarks to his own proof of this theorem (Crelle's Journal, vol. 24, 1842, pp. 96–99). This proof of Steiner is not only elegant and elementary but is has the additional merit of applying also to spherical triangles. However, from an elementary geometry standpoint it is not complete. For in the course of the proof it is required to prolong the side BC of a triangle DBC beyond C and to take a point E on this prolongation so that $DE+EC$ will have a given total length. No construction is given for E as is required in a geometry proof. This may be supplied without great difficulty in the case of a plane triangle, but this addition would complicate and lengthen the proof considerably. Another form of proof has been employed which depends upon the expression for the area of the general triangle in terms of the sides and the algebraic theorem that the arithmetic mean of any number of positive quantities is greater than their geometric mean. This furnishes a very brief and satisfactory proof, but it requires a proof of this algebraic theorem since it is not usually given in elementary texts.

Since it is quite likely that all students of plane geometry and many of their teachers have been led to believe that the theorem of this problem has been proved in their geometry texts, it would be desirable in such texts either to prove the theorem completely or to make clear just how much has been proved in the treatment given.

Also solved by H. C. BRADLEY, ALICE A. GRANT, MICHAEL GOLDBERG, S. B. LITTAUER, J. ROSENBAUM, and PAUL WERNICKE.

3253[1927, 217]. Proposed by N. A. Court, University of Oklahoma.

Through the vertices of a given tetrahedron, planes are drawn parallel to the opposite faces. Prove that the faces of the new tetrahedron thus obtained have for their centroids the corresponding vertices of the given tetrahedron.

Note: The new tetrahedron may, by analogy with the case in the plane, be called the "anticomplementary tetrahedron" of the given tetrahedron.

SOLUTION BY THEODORE BENNETT

Using tetrahedral coordinates, let the given tetrahedron have vertices P_i and altitudes h_i; let the areas of the faces be A_i, and let the volume be V. If we use this as a coordinate tetrahedron, any point in space has coordinates x_i connected by the identity

$$A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 3V, \tag{1}$$

the x's being the perpendicular distances from the faces of the tetrahedron to the point in question. We note that

$$A_1h_1=A_2h_2=A_3h_3=A_4h_4=3V. \tag{2}$$

The four planes through the vertices parallel to the opposite faces have the equations

$$x_1 = h_1,\ x_2= h_2,\ x_3 = h_3,\ x_4 = h_4. \tag{3}$$

From (1), (2), and (3) we find that the last three of these planes meet at the point

$$Q_1 = \left(\frac{3V - A_2h_2 - A_3h_3 - A_4h_4}{A_1}, h_2, h_3, h_4\right) = (-2h_1, h_2, h_3, h_4).$$

Similarly, the remaining vertices of the new tetrahedron are

$$Q_2, Q_3, Q_4 = (h_1, -2h_2, h_3, h_4),\ (h_1, h_2, -2h_3, h_4),\ (h_1, h_2, h_3, -2h_4).$$

As in Cartesian coordinates, the centroid of any triangle has for coordinates one third the sum of the corresponding coordinates of the vertices. Thus the centroid of $Q_1Q_2Q_3$ has coordinates $(0, 0, 0, h_4)$ and is therefore the point P_4. Similarly for the other faces.

Also solved by MICHAEL GOLDBERG, A. PELLETIER, J. ROSENBAUM, and PAUL WERNICKE.

A second solution by THEODORE BENNETT was received.

3256[1927, 217]. Proposed by C. N. Schmall, New York City.

Two spheres, with radii R and r, are inscribed in a right circular cone so that the greater touches the smaller and the base of the cone. If v is the volume of the cone, show that $v=2\pi R^5/3r(R-r)$.

SOLUTION BY D. LAWRENCE BARRICK, Montezuma College, Montezuma, N. M.

Let A be the vertex of the cone and AD its altitude. Then AD contains the centers O and H of the two inscribed spheres of radii r and R, respectively. A plane through AD cuts out the element AB and the radius DB of the base. Let OX and HW be the radii to the points of contact of the spheres with AB and let OK be perpendicular to HW at K. Then $HD=R$, $OH=R+r$, $HK=R-r$, and hence $OK=2(Rr)^{1/2}$. From similar triangles, we have $AH/R=(R+r)/(R-r)$; and hence $AD=2R^2/(R-r)$. Then $DB/AD=(R-r)/2(Rr)^{1/2}$; and hence $DB=R^2/(Rr)^{1/2}$. It now follows that $v=2\pi R^5/3r(R-r)$.

Also solved by R. P. AGNEW, E. F. ALLEN, THEODORE BENNETT, H. C. BRADLEY, W. B. CURRIE, L. A. DYE, PHILIP FITCH, J. S. GEORGES, MICHAEL GOLDBERG, ALICE A. GRANT, H. M. LUFKIN, R. E. MORRIS, R. M. MCFARLAND, NINA M. ALDERTON, and A. PELLETIER.

3260[1927, 272]. Proposed by B. C. Keeler, New York City.

Prove that the following series is convergent.

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots+\left\{\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right\}^{-n}+\cdots$$

SOLUTION BY LAURENCE HAMPTON, University of Oklahoma.

From the binomial expansion of $(1+n^{-1})^{n+1}$ we have

$$(1+n^{-1})^{n+1}-(1+n^{-1}) \geqq 1+(n+1)/2n > 3/2.$$

Hence the nth term of the series is less than $(2/3)^n$, and the sum of any number of terms is less than 2.

Also solved by THEODORE BENNETT, MICHAEL GOLDBERG, R. E. MORITZ, A. PELLETIER, and the PROPOSER.

3261[1927, 272]. Proposed by J. Rosenbaum, Milford, Connecticut.

On the sides AB, BC, CD, and DA of a quadrilateral $ABCD$, the points P, Q, R, and S are taken so that $AP/AB=BQ/BC=CR/CD=DS/DA$, thus forming the quadrilateral $PQRS$.

Prove that if the two quadrilaterals are similar then they are parallelograms.

SOLUTION BY PAUL WERNICKE, Washington, D. C.

The quadrilaterals $ABCD$ and $PQRS$ are assumed to be similar as written. Let $m=AP/AB$, and the vectors $AB=\beta$, $AD=\delta$, $AC=c\beta+k\delta$. Let the diagonals of $ABCD$ meet in F, those of $PQRS$, in V.

$$\text{Then } AF = (c\beta + k\delta)/(c + 1),\ BF = k(\delta - \beta)/(c + k) = k \cdot BD/(c + k).$$

On account of the similarity, V divides the diagonals

$$PR = AC + CR - AP = c\beta + k\delta + m(-c\beta - k\delta + \delta) - m\beta = (c - mc - m)\beta + (k - mk + m)\delta$$

$$\text{and } QS = AS - (AB + BQ) = (1 - m)\delta - [\beta + m(c\beta + k\delta - \beta)]$$

$$= (-1 + m - mc)\beta + (1 - m - mk)\delta$$

as AC and BD are divided in F. Therefore

$$PV = [(c - mc - m)\beta + (k - mk + m)\delta]/(c + k);$$

$$QV = [k(-1 + m - mc)\beta + k(1 - m - mk)\delta]/(c + k).$$

The relation $PB+BQ=PV+VQ$ can be put into the form

$$(1 - 2m + cm)\beta + km\delta = [(c + k - cm - km - m + ckm)\beta + m(1 + k^2)\delta]/(c + k)$$

whence

$$1 - 2m + cm = [c + k - cm - km - m + ckm]/(c + k),\ km = m(1 + k^2)/(c + k).$$

Discarding the trivial case $m=0$, we find $c=k=1$, or $c=-k=\pm i$. Thus, if the quadrilaterals are both real $c=k=1$, and they are both parallelograms.

Also solved by A. PELLETIER.

NOTE BY THE EDITORS: It can be shown further that a necessary and sufficient condition for the similarity of the quadrilaterals $ABCD$ and $PQRS$ (as written) is that they be squares.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Additional circulars of information concerning the Congress of Mathematicians to be held at Bologna next September, and blanks for abstracts of papers to be presented have been sent to all those whose names are on the list of prospective participants, as received at the office. Additional copies may be obtained upon request from the central office of the American Mathematical Society, 501 West 116th Street, New York City.

Dr. RALPH D. DONER and Dr. DUNCAN C. HARKIN have been made associate professors of mathematics at the Alabama Polytechnic Institute.

Professor GEORGE W. GORRELL has been made head of the department of mathematics at the University of Denver.

Miss MARY GERTRUDE HASEMAN has been made head of the department of mathematics and advisor of women at the Junior College of Connecticut, which opened its doors for the first time on February 1, 1928.

Professor GEORGE H. HENDERSON of Dalhousie University is on leave of absence. Mr. W. J. JACKSON and Mr. J. G. ADSHEAD have been made lecturers at this university.

Dr. M. R. RICHARDSON has been made assistant professor of mathematics at the Georgia School of Technology.

Professor J. E. ROWE of the College of William and Mary has been allowed two sets of letters patent on his trinometer and associated instruments by the United States Commissioner of Patents.

Mr. H. E. WAHLERT of the Georgia School of Technology resigned in January 1928. He has been succeeded by Mr. R. F. WATKINS.

Assistant Professor HERBERT H. WHEATON has been promoted to an associate professorship at Fresno State College.

The following appointments to instructorships in mathematics are announced:

Alabama Polytechnic Institute, Mr. L. L. GARNER, Mr. Z. M. PIRENIAN;

Colorado College, Miss MARTHA C. BELSCHNER, Miss HELEN E. VON BOSTON;

University of Denver, Mr. T. R. CUYKENDALL;

Fresno State College, Mrs. MABEL KELLY STURM;

Georgia School of Technology, Mr. ROY A. SMITH, Mr. CARL A. BENANDER.

The following courses in mathematics are announced for the summer of 1928.

University of Indiana. In addition to the usual courses in general mathematics, calculus, and mathematical theory of investment, the following advanced courses are offered. By Professor S. C. DAVISSON: Advanced differential calculus; Advanced algebra; Differential geometry. By Professor D. A. ROTHROCK: History of mathematics; Fourier series. By Professor K. P. WILLIAMS: Advanced analytic geometry; Differential equations.

University of Michigan, June 25 to August 17. In addition to courses in algebra, plane and solid geometry, trigonometry, analytic geometry, elementary calculus, mathematical statistics, and the mathematical theory of

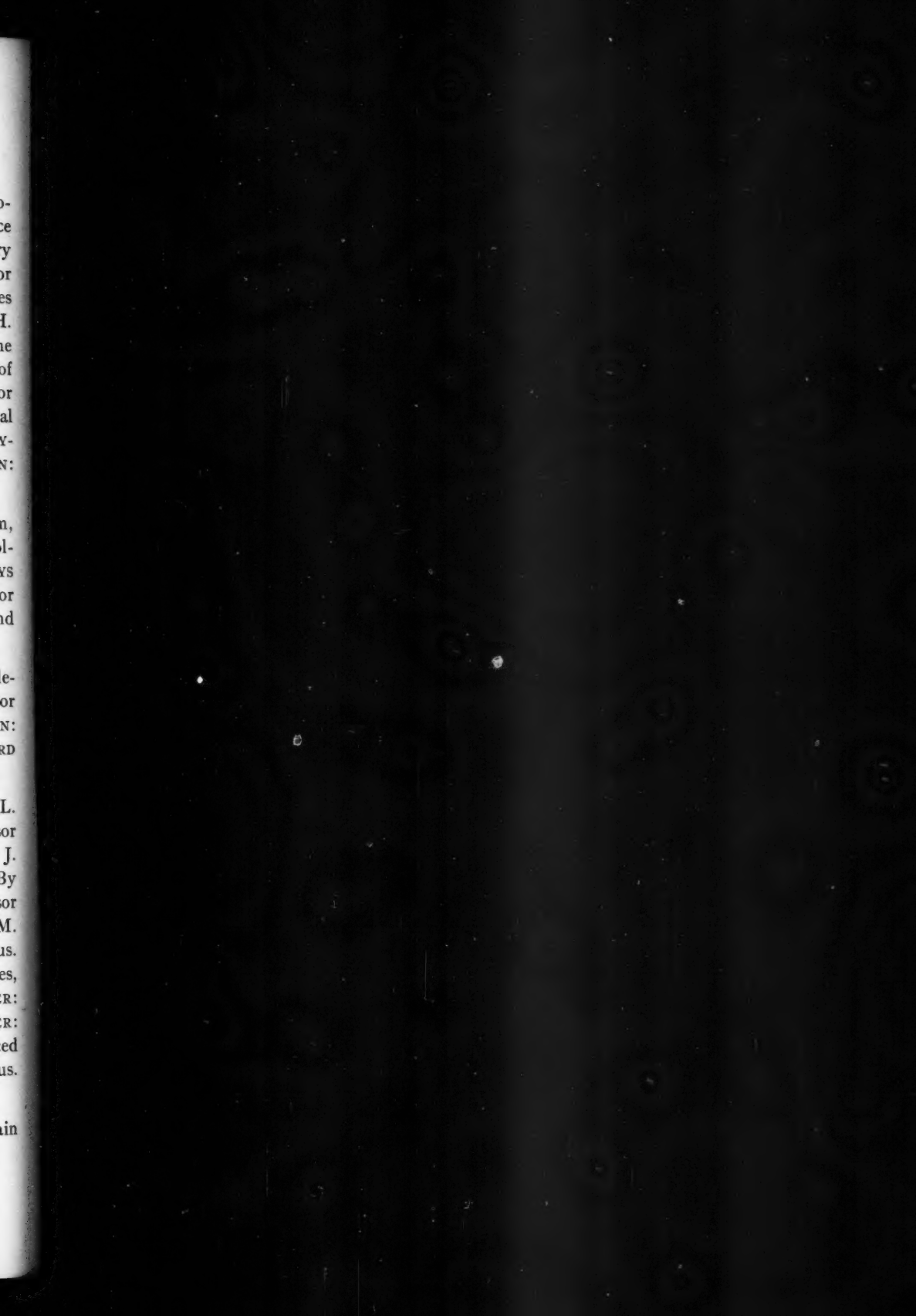
interest and insurance, the following advanced courses are offered. By Professor W. B. FORD; Advanced calculus; Infinite series with special reference to Fourier series. By Professor L. C. KARPINSKI: Teaching of algebra; History of mathematics. By Professor PETER FIELD: Vector analysis. By Professor T. R. RUNNING: Graphical methods. By Professor J. W. BRADSHAW: Figures of solid geometry; Selected topics in projective geometry. By Professor T. H. HILDEBRANDT: Theory of functions of a complex variable; Theory of the potential. By Professor H. C. CARVER: Advanced mathematical theory of statistics. By Professor L. A. HOPKINS: Elements of mechanics. By Professor C. J. COE: Integral equations. By Professor NORMAN ANNING: Differential equations; Determinants and theory of equations. By Professor J. A. NYSWANDER: Theory of probability; Finite differences. By Mr. O. J. PETERSON: Solid analytic geometry.

University of Minnesota, first term, June 16 to July 28; second term, July 30 to September 1. In addition to the usual elementary work, the following courses will be offered, First term: By Assistant Professor GLADYS GIBBENS: Differential equations; Synthetic metric geometry. By Professor W. L. HART: Advanced calculus; Reading in advanced mathematics. Second term: By Professor A. L. UNDERHILL: Reading in advanced mathematics.

University of Missouri, June 7 to August 3. In addition to the usual elementary courses, the following advanced courses are offered: By Professor INGOLD: Modern geometry; Selected topics from geometry. By Professor WAHLIN: Advanced algebra; Selected topics from algebra. By Professor L. R. FORD (Rice Institute): Calculus; Selected topics from analysis.

University of Texas, first term, June 5 to July 16. By Professor R. L. MOORE: Functions of real variables; Foundations of geometry. By Professor E. L. DODD: Infinite processes; Mathematical statistics. By Professor H. J. ETTLINGER: Boundary value problems; Ruler and compass constructions. By Professor C. D. RICE: Differential geometry; Advanced calculus. By Professor P. M. BATCHELDER: Teaching problems in mathematics. By Professor C. M. CLEVELAND: Advanced calculus. By Professor MARY DECHERD: Calculus. By Professor GOLDIE HORTON: Theory of equations. All freshman courses, both terms. Second term, July 16 to August 27. By Professor H. J. ETTLINGER: Boundary value problems; Definite integrals. By Professor P. M. BATCHELDER: Teaching problems in mathematics. By Professor C. M. CLEVELAND: Advanced calculus. By Professor R. G. LUBBEN: Non-Euclidean geometry; Calculus. By Professor G. T. WHYBURN: Functions of real variables; Calculus.

Professor H. E. RUSSELL of the University of Denver was killed by a train in May 1927.



USES OF PUNCHED CARD EQUIPMENT IN MATHEMATICS

By GEORGE W. SNEDECOR, Iowa State College

Two papers[1] recently appearing in the Monthly have served to acquaint its readers with some of the varied adaptations of the punched card sorting and tabulating machines, especially to accounting and to the compilation of statistics of population, insurance, and economics. In some institutions the equipment, being already installed, may be available for use by members of the mathematics departments. In other institutions the outfit might be secured, as it has been at Iowa State College, by the cooperation of several interested departments. It is the purpose of this article to explain in some detail how such an installation may be used by mathematicians, especially those interested in statistics.

The operation of the various units of the Hollerith equipment was fully described by Mr. Johns.[2] One feature only will be amplified, viz., the manner in which facts may be transferred to cards by punching.[3] For this purpose as many of the columns of a standard card as may be needed are grouped into "fields" of one or more columns each, one field for each variable. In the case of a non-numerical variable, such as "nationality" or "department of instruction," a "code" is adopted in which each individual is designated by a suitable number. Numerical data may also be coded, a process which will be discussed below. The number of columns required for a field is the same as the number

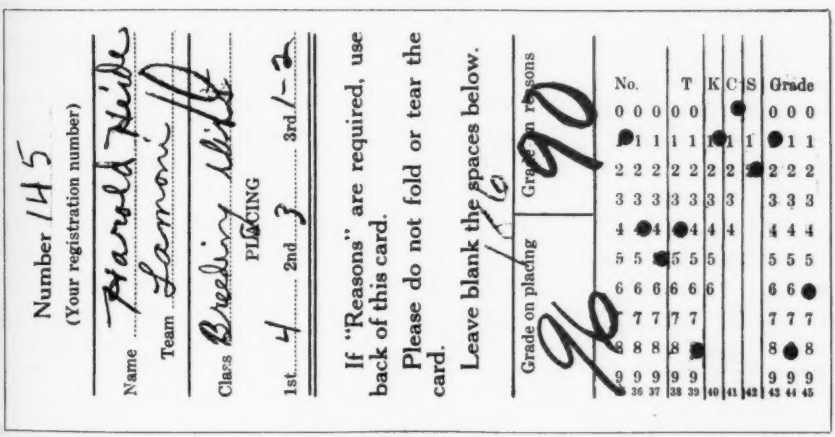


FIGURE 1

[1] Victor Johns, *On the Mechanical Handling of Statistics*, vol. 33, (1926), p. 494.
F. A. Harper and C. W. Vaughn, *Machines for Handling Statistics*, vol. 34 (1927), p. 449.

[2] Loc. cit.

[3] See also Edmund E. Day, *Statistical Analysis*, Macmillan, 1925, p. 394; and B. F. Young, *Statistics as Applied in Business*, Ronald Press, 1925, p. 493.